Exercise walk-through

Find the Area of a Polar Region or the Area Bounded by a Single Polar Curve ▪ 9.8

eXercise

You must be able to

- apply $A = \dfrac{1}{2}\displaystyle\int_{\alpha}^{\beta} \left(f(\theta)\right)^2 \, d\theta$
- choose the correct θ-limits that trace the region once
- set up the area for a full curve or a sector

Method — The polar area formula

The area swept by $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ is

$$A = \frac{1}{2} \int_{\alpha}^{\beta} \big(f(\theta)\big)^2 \, d\theta.$$

The region is a fan of thin triangular sectors of area $\frac{1}{2} r^2 \, d\theta$.

Method — A worked area

The circle $r = 2$: $A = \dfrac{1}{2}\displaystyle\int_0^{2\pi} 2^2\, d\theta = \dfrac{1}{2}(4)(2\pi) = 4\pi$ — matching πr^2.

Method — Choosing the limits

For $r = 2\cos\theta$ (a circle traced once as θ goes from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$), integrate over exactly that range so the curve is traced **once**.

Method — One petal of a rose

For $r = \sin(2\theta)$, one petal is traced as θ goes from 0 to $\frac{\pi}{2}$. Its area is

$$\frac{1}{2}\int_0^{\pi/2} \sin^2(2\theta)\, d\theta.$$

Practice 2.1 ▪ 1 mark

State the polar area formula.

Solution 2.1

Method: The polar area formula

Worked steps

$$A = \frac{1}{2}\int_{\alpha}^{\beta} \big(f(\theta)\big)^2 \, d\theta \quad \boxed{\textbf{B1}}.$$

Practice 2.2 ▪ 2 marks

Set up the area enclosed by $r = 3$ (a full circle).

Solution 2.2

Method: A worked area

Worked steps

$$A = \frac{1}{2}\int_0^{2\pi} 3^2 \, d\theta \quad \textbf{M1}\,\textbf{A1}.$$

Practice 2.3 ▪ 2 marks

Evaluate $\frac{1}{2}\int_0^{2\pi} 9\, d\theta$.

Solution 2.3

Worked steps

$\frac{1}{2}(9)(2\pi) = 9\pi$ **M1** **A1**.

Exam-style 3.1 ▪ 1 mark

The area of a polar region is

A $\displaystyle\int_{\alpha}^{\beta} f(\theta)\, d\theta$

B $\displaystyle\frac{1}{2}\int_{\alpha}^{\beta} \big(f(\theta)\big)^2\, d\theta$

C $\displaystyle\pi\int_{\alpha}^{\beta} f(\theta)\, d\theta$

D $\displaystyle\frac{1}{2}\int_{\alpha}^{\beta} f(\theta)\, d\theta$

Solution 3.1

Method: The polar area formula

A $\int_{\alpha}^{\beta} f(\theta)\, d\theta$

B $\frac{1}{2}\int_{\alpha}^{\beta} \left(f(\theta)\right)^2\, d\theta$ ✓

C $\pi\int_{\alpha}^{\beta} f(\theta)\, d\theta$

D $\frac{1}{2}\int_{\alpha}^{\beta} f(\theta)\, d\theta$

Worked steps

Exam-style 3.2 ▪ 2 marks

Consider $r = 1 + \cos\theta$ (a cardioid), traced once for $0 \leq \theta \leq 2\pi$.

(a) Set up the area integral.

(b) Expand $(1 + \cos\theta)^2$ ready to integrate.

Solution 3.2

Method: Choosing the limits

Worked steps

(a) $A = \dfrac{1}{2}\displaystyle\int_0^{2\pi} (1 + \cos\theta)^2 \, d\theta$ **M1** **A1**. (b) $(1 + \cos\theta)^2 = 1 + 2\cos\theta + \cos^2\theta$ **M1** **A1**.

Exam-style 3.3 ▪ 3 marks

One petal of the rose $r = 2\sin(3\theta)$ is traced as θ goes from 0 to $\frac{\pi}{3}$. Set up the area integral for one petal.

Solution 3.3

Method: One petal of a rose

Worked steps

$$A = \frac{1}{2}\int_0^{\pi/3} \big(2\sin(3\theta)\big)^2 \, d\theta = \frac{1}{2}\int_0^{\pi/3} 4\sin^2(3\theta) \, d\theta \quad \textbf{M1}\ \textbf{M1}\ \textbf{A1}.$$