Past-paper walk-through

Solving Motion Problems Using Parametric and Vector-Valued Functions ▪ 9.6

eXercise

Questions in this set

- 2024 Q2
- 2023 Q2
- 2022 Q2
- 2021 Q2
- 2018 Q2
- 2016 Q2
- 2015 Q2

Reading the mark scheme

- **B1** a mark that stands on its own – the point is either made or not.
- **M1** a *method* mark: the right approach, even if the number is wrong.
- **A1** an *answer* mark, and it usually needs its M mark first.
- **C1** a working mark on the way to the answer.
- // separates answers the examiner accepts equally.
- **ecf** = error carried forward: a later part is marked on your own earlier answer.
- **owtte** = "or words to that effect" – the wording need not match.

Question 2

2024 ▪ Q2

A particle moving along a curve in the xy-plane has position $(x(t), y(t))$ at time t seconds, where $x(t)$ and $y(t)$ are measured in centimeters. It is known that $x'(t) = 8t - t^2$ and $y'(t) = -t + \sqrt{t^{1.2} + 20}$. At time $t = 2$ seconds, the particle is at the point $(3, 6)$.

(a) Find the speed of the particle at time $t = 2$ seconds. Show the setup for your calculations.

(b) Find the total distance traveled by the particle over the time interval $0 \leq t \leq 2$. Show the setup for your calculations.

(c) Find the y-coordinate of the position of the particle at the time $t = 0$. Show the setup for your calculations.

Question 2

(d) For $2 \leq t \leq 8$, the particle remains in the first quadrant. Find all times t in the interval $2 \leq t \leq 8$ when the particle is moving toward the x-axis. Give a reason for your answer.

Solution 2

(a)

Mark scheme

- Speed $= \sqrt{(x'(2))^2 + (y'(2))^2}, where x'(t) = 8t - t^2 and y'(t) = -t + \sqrt{t^{1.2} + 20}. Evaluating at t = 2 gives speed \approx 12.3048506 \approx 12.305 (or 12.304) centimeters per second. Points : 1 for the correct setup \sqrt{(x'(2))^2 + (y'(2))^2} (or the equivalent general expression evaluated at t = 2), 1 for the correct numerical answer 12.305 (or 12.304); missing/incorrect units do not affect scoringh$

Solution 2

(b)

Mark scheme

- Total distance $= \int_0^2 \sqrt{(x'(t))^2 + (y'(t))^2}\, dt = 15.901715 \approx 15.902 (or 15.901) centimeters. Points : 1 forthecorrectintegralsetup \int_0^2 \sqrt{(x'(t))^2 + (y'(t))^2}\, dt (withorwithoutthedifferential), 1 forthec$

Solution 2

(c)

Mark scheme

- $y(0) = 6 + \int_2^0 y'(t)\,dt = 6 - 7.173613 = -1.173613 \approx -1.174 (or - 1.173). Points:$
$1 foracorrectdefiniteintegralofy'(t)(limits2to0, or0to2), 1 forcorrectlycombiningthatintegralwit$
$6(addedorsubtractedconsistentlywiththelimitsused), 1 forthecorrectfinalnumericanswer, contir$

Solution 2

(d) Mark scheme

- Since the particle stays in the first quadrant (y(t)>0) for 2$\leq t \leq 8, itmovestowardthex-axisexactlywheny'(t) < 0, i.e.when-t+\sqrt{t^{1.2}+20} < 0. Solvingy'(t) = 0onthisintervalgivest \approx 5.222(or5.221), sotheparticlemovestowardthex-axisfor5.222 < t \leq 8(intervalcanbeopen, closed, orhalf-open). Points: 1forconsideringthesignofy'(t)(statingy'(t) = 0, y'(t) < 0, y'(t) > 0, orthecriticalvaluet = 5.222), 1forthecorrectintervaltogetherwithexplicitreasoningthaty'(t) < 0there(thispointcannotbeearnedwithoutthefirst).$

Question 2

2023 ▪ Q2

[Graph of a curve in the xy-plane; x-axis from O to 5, y-axis with gridlines at 1 and 2. The curve starts at the origin $(0, 0)$, rises as a straight line through about $(1, 1)$ up to a peak near $(4, 2)$, then curves down steeply to end at $(5, 0)$.]

For $0 \leq t \leq \pi$, a particle is moving along the curve shown so that its position at time t is $(x(t), y(t))$, where $x(t)$ is not explicitly given and $y(t) = 2 \sin t$. It is known that $\dfrac{dx}{dt} = e^{\cos t}$. At time $t = 0$, the particle is at position $(1, 0)$.

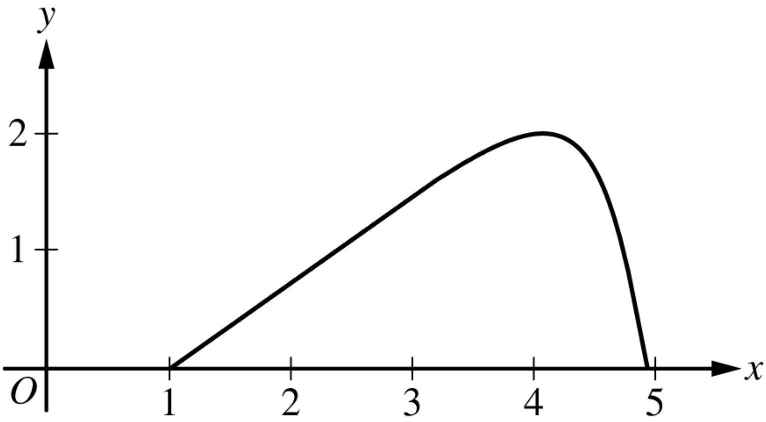
(a) Find the acceleration vector of the particle at time $t = 1$. Show the setup for your calculations.

(b) For $0 \leq t \leq \pi$, find the first time t at which the speed of the particle is 1.5. Show the work that leads to your answer.

Question 2

(c) Find the slope of the line tangent to the path of the particle at time $t = 1$. Find the x-coordinate of the position of the particle at time $t = 1$. Show the work that leads to your answers.
(d) Find the total distance traveled by the particle over the time interval $0 \leq t \leq \pi$. Show the setup for your calculations.

Question 2



Solution 2

(a)

Mark scheme

- $x''(t) = \frac{d}{dt}(e^{\cos t}) = -e^{\cos t}\sin t$, so $x''(1) = -1.444407$. Since $y(t) = 2\sin t$, $y'(t) = 2\cos t$, $y''(t) = -2\sin t$, so $y''(1) = -1.682942$. The acceleration vector at $t = 1$ is $a(1) = \langle -1.444, -1.683\rangle$ (or $\langle -1.444, -1.682\rangle$). (1 point for $x''(1)$ with setup; 1 point for $y''(1)$ with setup — an unsupported correct vector earns only 1 of the 2 points.)

Solution 2

(b)

Mark scheme

- Speed $= \sqrt{(dx/dt)^2 + (dy/dt)^2} = \sqrt{(e^{\cos t})^2 + (2\cos t)^2}$. Setting speed $= 1.5$ for $0 \leq t \leq \pi$: $t = 1.254472$ or $t = 2.358077$. The first time the speed of the particle is 1.5 is $t = 1.254$. (1 point for setting up the speed equation equal to 1.5; 1 point for the correct first-time answer $t = 1.254$ — the response need not consider $t = 2.358077$.)

Solution 2

(c) Mark scheme

- $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2\cos t}{e^{\cos t}}$, so at $t = 1$: $\left.\frac{dy}{dx}\right|_{t=1} = \frac{2\cos 1}{e^{\cos 1}} = 0.629530 \approx 0.630$ (or 0.629). For the x-coordinate: $x(1) = x(0) + \int_0^1 \frac{dx}{dt}\,dt = 1 + \int_0^1 e^{\cos t}\,dt = 3.341575 \approx 3.342$ (or 3.341). (1 point for the slope with supporting work, communicating $dy/dx = (dy/dt)/(dx/dt)$; 1 point for presenting the definite integral $\int_0^1 e^{\cos t}\,dt$ with or without the initial condition; 1 point for the correct value of $x(1)$.)

Solution 2

(d)

Mark scheme

- Total distance $= \int_0^{\pi} \sqrt{(e^{\cos t})^2 + (2\cos t)^2}\, dt = 6.034611 \approx 6.035$ (or 6.034). (1 point for presenting the correct speed integrand — from part (b) — in a definite integral; 1 point for the correct numeric answer; an unsupported answer of 6.035 alone earns neither point.)

Question 2

2022 ▪ Q2

A particle moving along a curve in the xy-plane is at position $(x(t), y(t))$ at time $t > 0$. The particle moves in such a way that $\dfrac{dx}{dt} = \sqrt{1+t^2}$ and $\dfrac{dy}{dt} = \ln(2+t^2)$. At time $t = 4$, the particle is at the point $(1, 5)$.

(a) Find the slope of the line tangent to the path of the particle at time $t = 4$.

(b) Find the speed of the particle at time $t = 4$, and find the acceleration vector of the particle at time $t = 4$.

(c) Find the y-coordinate of the particle's position at time $t = 6$.

(d) Find the total distance the particle travels along the curve from time $t = 4$ to time $t = 6$.

Solution 2

(a) Mark scheme

- $\left.\dfrac{dy}{dx}\right|_{t=4} = \dfrac{y'(4)}{x'(4)} = \dfrac{\ln(2+4^2)}{\sqrt{1+4^2}} = \dfrac{\ln 18}{\sqrt{17}} = 0.701018 \approx 0.701$. 1 point for the correct numeric slope, with the setup (quotient of $y'(4)$ and $x'(4)$) evident in the response.

Solution 2

(b) Mark scheme

- Speed $= \sqrt{(x'(4))^2 + (y'(4))^2} = \sqrt{17 + (\ln 18)^2} = 5.035300 \approx 5.035$ (1 point).
  Acceleration: $x''(t) = \dfrac{t}{\sqrt{1+t^2}}$, $y''(t) = \dfrac{2t}{2+t^2}$, so
  $a(4) = \langle x''(4), y''(4)\rangle = \left\langle \dfrac{4}{\sqrt{17}}, \dfrac{4}{9} \right\rangle = \langle 0.970143, 0.444444\rangle \approx \langle 0.970, 0.444\rangle$ (1 point for each component, earned independently — if the components are reversed, neither point is earned).

Solution 2

(c)

Mark scheme

- $y(6) = y(4) + \int_4^6 \ln(2+t^2)\,dt = 5 + 6.570517 = 11.570517 \approx 11.571$ (or 11.570). 3 points: 1 for the integrand $\ln(2+t^2)$ appearing in an integral, 1 for adding the correct definite integral to $y(4) = 5$, 1 for the final numeric answer.

Solution 2

(d)

Mark scheme

- Total distance $= \int_4^6 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt =$ $\int_4^6 \sqrt{(1+t^2) + (\ln(2+t^2))^2}\, dt = 12.136228 \approx 12.136$. 2 points: 1 for the correct integrand in a definite integral from $t = 4$ to $t = 6$, 1 for the correct numeric answer (an unsupported answer of 12.136 earns neither point).

Question 2

2021 ▪ Q2

For time $t \geq 0$, a particle moves in the xy-plane with position $(x(t), y(t))$ and velocity vector $\left\langle (t-1)e^{t^2}, \sin\left(t^{1.25}\right) \right\rangle$. At time $t = 0$, the position of the particle is $(-2, 5)$.

(a) Find the speed of the particle at time $t = 1.2$. Find the acceleration vector of the particle at time $t = 1.2$.

(b) Find the total distance traveled by the particle over the time interval $0 \leq t \leq 1.2$.

(c) Find the coordinates of the point at which the particle is farthest to the left for $t \geq 0$. Explain why there is no point at which the particle is farthest to the right for $t \geq 0$.

Solution 2

(a)

Mark scheme

- Speed $= \sqrt{(x'(1.2))^2 + (y'(1.2))^2} \approx 1.271488$, so speed ≈ 1.271. Acceleration vector $= \langle x''(1.2), y''(1.2)\rangle \approx \langle 6.247, 0.405\rangle$ (or 6.246). (1 pt: correct speed; 1 pt: correct acceleration vector — both require a supported numeric computation, not just a bare final answer.)

Solution 2

(b)

Mark scheme

- Total distance $= \int_0^{1.2} \sqrt{(x'(t))^2 + (y'(t))^2}\, dt \approx 1.009817$, so distance ≈ 1.010 (or 1.009). (1 pt: presenting the correct integrand $\sqrt{(x'(t))^2 + (y'(t))^2}$ in a definite integral with limits 0 to 1.2; 1 pt: correct numeric answer.)

Solution 2

(c) Mark scheme

- $x'(t) = (t-1)e^{t^2} = 0 \Rightarrow t = 1$ is the only critical point (since $e^{t^2} > 0$ always). Because $x'(t) < 0$ for $0 < t < 1$ and $x'(t) > 0$ for $t > 1$, the particle is farthest left at $t = 1$. $x(1) = -2 + \int_0^1 x'(t)\,dt \approx -2.603511$ and $y(1) = 5 + \int_0^1 y'(t)\,dt \approx 5.410486$, so the leftmost point is $\approx (-2.604, 5.410)$ (or -2.603). Since $x'(t) > 0$ for all $t > 1$, the particle moves continuously to the right forever after $t = 1$ (and $x(2) > x(0)$, confirming motion extends right of the initial position), so there is no point at which $x(t)$ attains a maximum — hence no farthest-right point exists. (1 pt: sets $x'(t) = 0$; 1 pt: valid reasoning that the particle is leftmost at $t = 1$ from the sign change of x'; 1 pt: one coordinate of the leftmost position with supporting integral; 1 pt: complete leftmost position

Solution 2

$(-2.604, 5.410)$; 1 pt: valid explanation that $x'(t) > 0$ for $t > 1$ so motion continues right without bound, hence no farthest-right point.)

Question 2

2018 ▪ Q2

Researchers on a boat are investigating plankton cells in a sea. At a depth of h meters, the density of plankton cells, in millions of cells per cubic meter, is modeled by $p(h) = 0.2h^2 e^{-0.0025h^2}$ for $0 \leq h \leq 30$ and is modeled by $f(h)$ for $h \geq 30$. The continuous function f is not explicitly given.

(a) Find $p'(25)$. Using correct units, interpret the meaning of $p'(25)$ in the context of the problem.

Question 2

2018 ▪ Q2

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(b) Consider a vertical column of water in this sea with horizontal cross sections of constant area 3 square meters. To the nearest million, how many plankton cells are in this column of water between $h = 0$ and $h = 30$ meters?

Question 2

2018 ▪ Q2

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(c) There is a function u such that $0 \le f(h) \le u(h)$ for all $h \ge 30$ and $\int_{30}^{\infty} u(h)\, dh = 105$. The column of water in part (b) is K meters deep, where $K > 30$. Write an expression involving one or more integrals that gives the number of plankton cells, in millions, in the entire column. Explain why the number of plankton cells in the column is less than or equal to 2000 million.

Question 2

2018 ▪ Q2

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(d) The boat is moving on the surface of the sea. At time $t \geq 0$, the position of the boat is $(x(t), y(t))$, where $x'(t) = 662\sin(5t)$ and $y'(t) = 880\cos(6t)$. Time t is measured in hours, and $x(t)$ and $y(t)$ are measured in meters. Find the total distance traveled by the boat over the time interval $0 \leq t \leq 1$.

Solution 2

(a)

Mark scheme

- $p'(25) = -1.179$. At a depth of 25 meters, the density of plankton cells is changing at a rate of -1.179 million cells per cubic meter per meter. Points: 1 for the correct derivative value, 1 for the correct meaning stated with correct units (million cells per cubic meter per meter, decreasing).

Solution 2

(b)

Mark scheme

- $\int_0^{30} 3p(h)\,dh = 1675.414936$. There are approximately 1675 million plankton cells in the column of water between $h = 0$ and $h = 30$ meters. Points: 1 for the correct integrand $3p(h)$ (cross-sectional area 3 times density), 1 for the correct numeric answer (1675, to the nearest million).

Solution 2

(c) Mark scheme

- $\int_{30}^{K} 3f(h)\,dh$ represents the number of plankton cells, in millions, in the column of water from depth 30 meters to depth K meters. The number of plankton cells, in millions, in the entire column of water is $\int_0^{30} 3p(h)\,dh + \int_{30}^{K} 3f(h)\,dh$. Because $0 \le f(h) \le u(h)$ for all $h \ge 30$:
$3\int_{30}^{K} f(h)\,dh \le 3\int_{30}^{K} u(h)\,dh \le 3\int_{30}^{\infty} u(h)\,dh = 3 \cdot 105 = 315$. So the total number of plankton cells in the column is bounded by $1675.415 + 315 = 1990.415 \le 2000$ million. Points: 1 for the integral expression combining the part (b) integral with $\int_{30}^{K} 3f(h)\,dh$, 1 for comparing to the improper integral bound ($\le 3\int_{30}^{\infty} u(h)\,dh = 315$), 1 for the explanation completing the bound (1675.415+315 $\le$ 2000).

Solution 2

(d)

Mark scheme

- Total distance $= \int_0^1 \sqrt{(x'(t))^2 + (y'(t))^2}\, dt = 757.455862$ meters (accept 757.456 or 757.455). Points: 1 for the correct arc-length integrand using $x'(t) = 662\sin(5t)$ and $y'(t) = 880\cos(6t)$, 1 for the correct total distance.

Question 2

2016 ▪ Q2

At time t, the position of a particle moving in the xy-plane is given by the parametric functions $(x(t), y(t))$, where $\frac{dx}{dt} = t^2 + \sin(3t^2)$. The graph of y, consisting of three line segments, is shown in the figure above. At $t = 0$, the particle is at position $(5, 1)$. [Graph of $y(t)$: piecewise-linear, three line segments, drawn on axes with t from 0 to 4 and $y(t)$ from -2 to 2. Segment 1 goes from $(0, 1)$ down to $(1, -1)$. Segment 2 is horizontal from $(1, -1)$ to $(2, -1)$. Segment 3 goes from $(2, -1)$ up to $(4, 1)$.]

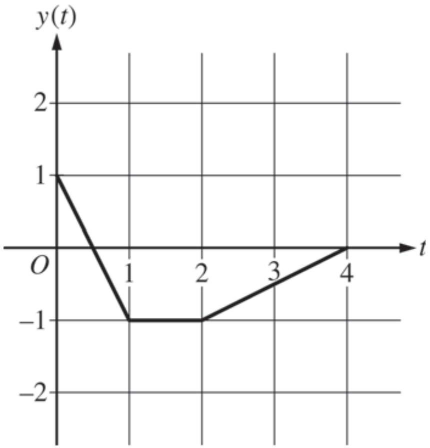
(a) Find the position of the particle at $t = 3$.

(b) Find the slope of the line tangent to the path of the particle at $t = 3$.

(c) Find the speed of the particle at $t = 3$.

(d) Find the total distance traveled by the particle from $t = 0$ to $t = 2$.

Question 2



Solution 2

(a)

Mark scheme

- $x(3) = x(0) + \int_0^3 x'(t)\,dt = 5 + 9.377035 \approx 14.377$, using $x'(t) = t^2 + \sin(3t^2)$ and the initial condition $x(0) = 5$. $y(3) = -0.5$, read from the piecewise-linear graph of y (the segment through $t = 3$ runs from $(2, -1)$ to $(4, 0)$, slope 0.5, so $y(3) = -1 + 0.5(1) = -0.5$). The position of the particle at $t = 3$ is $(14.377, -0.5)$. Award 1 point for the integral expression for $x(3)$, 1 point for using the initial condition $x(0) = 5$, and 1 point for the correct answer $(14.377, -0.5)$.

Solution 2

(b)

Mark scheme

- Slope $= \dfrac{y'(3)}{x'(3)} = \dfrac{0.5}{3^2 + \sin(27)} = \dfrac{0.5}{9.956376} \approx 0.05$, where $y'(3) = 0.5$ is the slope of the line segment of the $y(t)$ graph containing $t = 3$. Award 1 point for the correct slope ≈ 0.05.

Solution 2

(c)

Mark scheme

- Speed $= \sqrt{(x'(3))^2 + (y'(3))^2} = \sqrt{9.956376^2 + 0.5^2} \approx 9.969$ (or 9.968). Award 1 point for the correct expression for speed $\sqrt{(x'(t))^2 + (y'(t))^2}$ evaluated at $t = 3$, and 1 point for the numeric answer.

Solution 2

(d) Mark scheme

- Distance
  $= \int_0^2 \sqrt{(x'(t))^2 + (y'(t))^2}\, dt = \int_0^1 \sqrt{(x'(t))^2 + (-2)^2}\, dt + \int_1^2 \sqrt{(x'(t))^2 + 0^2}\, dt,$
  splitting at $t = 1$ because $y'(t) = -2$ on $[0, 1]$ (segment from $(0, 1)$ to $(1, -1)$)
  and $y'(t) = 0$ on $[1, 2]$ (flat segment from $(1, -1)$ to $(2, -1)$).
  $= 2.237871 + 2.112003 = 4.350$ (or 4.349). Award 1 point for the correct expression for distance, 1 point for correctly splitting into the two integrals with the right $y'(t)$ pieces, and 1 point for the numeric answer.

Question 2

2015 ▪ Q2

At time $t \geq 0$, a particle moving along a curve in the xy-plane has position $(x(t), y(t))$ with velocity vector $v(t) = \left(\cos(t^2),\ e^{0.5t}\right)$. At $t = 1$, the particle is at the point $(3, 5)$.

(a) Find the x-coordinate of the position of the particle at time $t = 2$.

(b) For $0 < t < 1$, there is a point on the curve at which the line tangent to the curve has a slope of 2. At what time is the object at that point?

(c) Find the time at which the speed of the particle is 3.

(d) Find the total distance traveled by the particle from time $t = 0$ to time $t = 1$.

Solution 2

(a)

Mark scheme

- $x(2) = 3 + \int_1^2 \cos(t^2)\,dt = 2.557$ (or 2.556). Award 1 point for the correct integral setup, 1 point for using the initial condition $x(1) = 3$, and 1 point for the correct final answer.

Solution 2

(b)

Mark scheme

- $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{e^{0.5t}}{\cos(t^2)}$. Setting $\frac{e^{0.5t}}{\cos(t^2)} = 2$ gives $t = 0.840$. Award 1 point for the correct slope expression in terms of t, and 1 point for the correct answer.

Solution 2

(c)

Mark scheme

- Speed $= \sqrt{\cos^2(t^2) + e^t}$. Setting $\sqrt{\cos^2(t^2) + e^t} = 3$ gives $t = 2.196$ (or 2.195). Award 1 point for the correct speed expression in terms of t, and 1 point for the correct answer.

Solution 2

(d)

Mark scheme

- Distance $= \int_0^1 \sqrt{\cos^2(t^2) + e^t}\, dt = 1.595$ (or 1.594). Award 1 point for the correct integral setup, and 1 point for the correct numeric answer.