Exercise walk-through

Solving Motion Problems Using Parametric and Vector-Valued Functions ▪ 9.6

eXercise

You must be able to

- find velocity, acceleration, and **speed** from a position vector
- find the **position at a later time** (start $+ \int \vec{v}\, dt$)
- find the **total distance travelled** $\int_a^b \sqrt{x'^2 + y'^2}\, dt$

Method — The motion toolkit

For a particle at $\vec{r}(t) = \langle x(t), y(t) \rangle$: velocity $\vec{v} = \langle x', y' \rangle$, acceleration $\vec{a} = \langle x'', y'' \rangle$, and speed $|\vec{v}| = \sqrt{x'^2 + y'^2}$.

Method — Position at a later time

If you know $\vec{r}(a)$ and $\vec{v}(t)$, the position at b is

$$\vec{r}(b) = \vec{r}(a) + \int_a^b \vec{v}(t)\, dt,$$

component by component.

Method — Total distance travelled

The **distance travelled** (not displacement) over $[a, b]$ is

$$\int_a^b \sqrt{\big(x'(t)\big)^2 + \big(y'(t)\big)^2}\, dt$$

— the parametric arc length of the path.

Method — A worked speed

For $\vec{v}(t) = \langle 3t, 4t \rangle$: speed $= \sqrt{9t^2 + 16t^2} = 5t$; at $t = 2$, speed $= 10$.

Practice 2.1 ▪ 1 mark

For $\vec{v}(t) = \langle 6, 8 \rangle$, find the speed.

Solution 2.1

Worked steps

$\sqrt{36+64} = 10$ B1.

Practice 2.2 ▪ 2 marks

A particle has $\vec{v}(t) = \langle 2t, 2 \rangle$. Set up the total distance travelled over $0 \leq t \leq 3$.

Solution 2.2

Method: Total distance travelled

Worked steps

$$\int_0^3 \sqrt{(2t)^2 + 2^2}\, dt = \int_0^3 \sqrt{4t^2 + 4}\, dt \quad \text{M1} \; \text{A1}.$$

Practice 2.3 ▪ 3 marks

A particle starts at $\vec{r}(0) = \langle 1, 2 \rangle$ with $\vec{v}(t) = \langle 4, 6t \rangle$. Find $\vec{r}(t)$.

Solution 2.3

Method: The motion toolkit

Worked steps

$\vec{r}(t) = \langle 4t + 1, 3t^2 + 2 \rangle$ M1 M1 A1.

Exam-style 3.1 ▪ 1 mark

The total distance a particle travels over $[a, b]$ is

A $\displaystyle\int_a^b \vec{v}\,dt$

B $|\vec{r}(b) - \vec{r}(a)|$

C $\displaystyle\int_a^b \sqrt{x'^2 + y'^2}\,dt$

D $\vec{r}(b) - \vec{r}(a)$

Solution 3.1

Method: Total distance travelled

A $\displaystyle\int_a^b \vec{v}\,dt$

B $|\vec{r}(b) - \vec{r}(a)|$

C $\displaystyle\int_a^b \sqrt{x'^2 + y'^2}\,dt$ ✓

D $\vec{r}(b) - \vec{r}(a)$

Worked steps

C — total distance is the arc-length integral of speed.

Exam-style 3.2 ▪ 3 marks

A particle moves with $\vec{v}(t) = \langle 3t^2, 2t \rangle$, starting at $\vec{r}(0) = \langle 0, 1 \rangle$.
(a) Find the position $\vec{r}(t)$.
(b) Find the speed at $t = 1$.

Solution 3.2

Method: The motion toolkit

Worked steps

(a) $\vec{r}(t) = \langle t^3, t^2 + 1 \rangle$ M1 M1 A1. (b) $\vec{v}(1) = \langle 3, 2 \rangle$; speed $= \sqrt{9+4} = \sqrt{13}$ M1 A1.

Exam-style 3.3 ▪ 3 marks

A particle has $\vec{v}(t) = \langle \cos t, \sin t \rangle$. Set up and evaluate the total distance travelled over $0 \le t \le \pi$.

Solution 3.3

Method: Total distance travelled

Worked steps

speed $= \sqrt{\cos^2 t + \sin^2 t} = 1$ M1; distance $= \int_0^{\pi} 1 \, dt = \pi$ M1 A1.