Exercise walk-through

Defining and Differentiating Vector-Valued Functions ▪ 9.4

eXercise

You must be able to

- read a **vector-valued function 向量值函数** $\vec{r}(t) = \langle x(t), y(t) \rangle$
- differentiate **component-wise** to get velocity and acceleration
- find the **speed** $|\vec{v}| = \sqrt{x'^2 + y'^2}$

Method — Position, velocity, acceleration

A vector function gives a position for each t: $\vec{r}(t) = \langle x(t), y(t)\rangle$. Differentiate each component:

$$\vec{v}(t) = \langle x'(t), y'(t)\rangle, \qquad \vec{a}(t) = \langle x''(t), y''(t)\rangle.$$

Method — A worked derivative

For $\vec{r}(t) = \langle t^2, \sin t \rangle$: $\vec{v}(t) = \langle 2t, \cos t \rangle$ and $\vec{a}(t) = \langle 2, -\sin t \rangle$.

Method — Speed is a magnitude

The **speed** is the length of the velocity vector:

$$|\vec{v}| = \sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2}.$$

For $\vec{v} = \langle 3, 4 \rangle$, speed $= \sqrt{9 + 16} = 5$.

Method — Evaluating at a point

At a specific t, plug in to get the numeric velocity, acceleration, and speed. At $t = 0$ for $\vec{r} = \langle t^2, \sin t\rangle$: $\vec{v}(0) = \langle 0, 1\rangle$, speed $= 1$.

Practice 2.1 ▪ 2 marks

For $\vec{r}(t) = \langle t^3, t^2 \rangle$, find $\vec{v}(t)$.

Solution 2.1

Worked steps

$\vec{v}(t) = \langle 3t^2, 2t \rangle$ M1 A1.

Practice 2.2 ▪ 2 marks

For $\vec{v}(t) = \langle 2t, 3 \rangle$, find the speed at $t = 2$.

Solution 2.2

Worked steps

$\vec{v}(2) = \langle 4, 3 \rangle$; speed $= \sqrt{16+9} = 5$ **M1** **A1**.

Practice 2.3 ▪ 2 marks

For $\vec{r}(t) = \langle \cos t, \sin t \rangle$, find $\vec{a}(t)$.

Solution 2.3

Worked steps

$\vec{v} = \langle -\sin t, \cos t \rangle$; $\vec{a} = \langle -\cos t, -\sin t \rangle$ M1 A1.

Exam-style 3.1 ▪ 1 mark

The speed of a particle with velocity $\vec{v} = \langle x', y' \rangle$ is

A $x' + y'$

B $\sqrt{x'^2 + y'^2}$

C $\langle x', y' \rangle$

D $x' y'$

Solution 3.1

A $x' + y'$

B $\sqrt{x'^2 + y'^2}$ ✓

C $\langle x', y' \rangle$

D $x'y'$

Worked steps

B — speed is the magnitude $\sqrt{x'^2 + y'^2}$.

Exam-style 3.2 ▪ 2 marks

A particle has position $\vec{r}(t) = \langle t^2 - 1, 2t \rangle$.
(a) Find $\vec{v}(t)$ and $\vec{a}(t)$.
(b) Find the speed at $t = 1$.

Solution 3.2

Worked steps

(a) $\vec{v} = \langle 2t, 2\rangle$, $\vec{a} = \langle 2, 0\rangle$ **M1** **A1**. (b) $\vec{v}(1) = \langle 2, 2\rangle$; speed $= \sqrt{4+4} = 2\sqrt{2}$ **M1** **A1**.

Exam-style 3.3 ▪ 4 marks

For $\vec{r}(t) = \langle e^t, e^{-t} \rangle$, find the velocity and speed at $t = 0$.

Solution 3.3

Worked steps

$\vec{v} = \langle e^t, -e^{-t} \rangle$ **M1**; at $t = 0$, $\vec{v} = \langle 1, -1 \rangle$ **M1**; speed $= \sqrt{1+1} = \sqrt{2}$ **M1** **A1**.