Exercise walk-through

Finding Arc Lengths of Curves Given by Parametric Equations ▪ 9.3

eXercise

You must be able to

- apply $L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$
- set up the integral from parametric equations
- recognise it as summing tiny hypotenuses $\sqrt{dx^2 + dy^2}$

Method — The parametric arc-length formula

For $x = x(t)$, $y = y(t)$ with t from a to b:

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Method — Setting it up

For $x = t^2$, $y = t^3$ on $[0, 1]$: $\dfrac{dx}{dt} = 2t$, $\dfrac{dy}{dt} = 3t^2$, so

$$L = \int_0^1 \sqrt{4t^2 + 9t^4}\, dt.$$

Method — A curve that simplifies

For a circle $x = \cos t$, $y = \sin t$: $\dfrac{dx}{dt} = -\sin t$, $\dfrac{dy}{dt} = \cos t$, so the integrand is $\sqrt{\sin^2 t + \cos^2 t} = 1$. Then $L = \int_0^{2\pi} 1\, dt = 2\pi$ (the circumference).

Method — Same idea as before

This is the $\sqrt{dx^2 + dy^2}$ hypotenuse summed along the curve, now with both x and y depending on t.

Practice 2.1 ▪ 1 mark

State the parametric arc-length formula.

Solution 2.1

Method: The parametric arc-length formula

Worked steps

$$L = \int_a^b \sqrt{(dx/dt)^2 + (dy/dt)^2}\, dt \quad \boxed{\textbf{B1}}.$$

Practice 2.2 ▪ 2 marks

For $x = t^2$, $y = t^3$, write $\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2$.

Solution 2.2

Method: The parametric arc-length formula

Worked steps

$(2t)^2 + (3t^2)^2 = 4t^2 + 9t^4$ M1 A1.

Practice 2.3 ▪ 3 marks

For $x = 3t$, $y = 4t$ on $[0, 2]$, set up and evaluate the arc length.

Solution 2.3

Worked steps

$\sqrt{9+16} = 5$; $L = \int_0^2 5\,dt = 10$ **M1** **M1** **A1**.

Exam-style 3.1 ▪ 1 mark

The parametric arc length uses the integrand

A $\sqrt{1 + (dy/dx)^2}$

B $\sqrt{(dx/dt)^2 + (dy/dt)^2}$

C $\dfrac{dy/dt}{dx/dt}$

D $(dx/dt)(dy/dt)$

Solution 3.1

Method: The parametric arc-length formula

A $\sqrt{1 + (dy/dx)^2}$

B $\sqrt{(dx/dt)^2 + (dy/dt)^2}$ ✓

C $\dfrac{dy/dt}{dx/dt}$

D $(dx/dt)(dy/dt)$

Worked steps

B — the parametric integrand is $\sqrt{(dx/dt)^2 + (dy/dt)^2}$.

Exam-style 3.2 ▪ 2 marks

A curve is $x = \cos t$, $y = \sin t$ for $0 \le t \le \pi$.
(a) Show the integrand simplifies to 1.
(b) Evaluate the arc length.

Solution 3.2

Method: A curve that simplifies

Worked steps

(a) $(-\sin t)^2 + (\cos t)^2 = \sin^2 t + \cos^2 t = 1$ **M1** **A1**. (b) $\int_0^\pi 1\, dt = \pi$ **B1**.

Exam-style 3.3 ▪ 3 marks

Set up (do not evaluate) the arc-length integral for $x = t - \sin t$, $y = 1 - \cos t$ on $0 \le t \le 2\pi$.

Solution 3.3

Worked steps

$$\frac{dx}{dt} = 1 - \cos t,\ \frac{dy}{dt} = \sin t \ \textbf{M1};\ L = \int_0^{2\pi} \sqrt{(1-\cos t)^2 + \sin^2 t}\, dt \ \textbf{M1}\ \textbf{A1}.$$