Exercise walk-through

Second Derivatives of Parametric Equations ▪ 9.2

eXercise

You must be able to

- compute $\dfrac{d^2y}{dx^2} = \dfrac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt}$
- avoid the wrong "divide the second derivatives" shortcut
- use the sign of $\dfrac{d^2y}{dx^2}$ to state **concavity 凹凸性**

Method — The correct formula

The second derivative is **not** $\dfrac{d^2y/dt^2}{d^2x/dt^2}$. Instead, differentiate the first derivative with respect to t, then divide by $\dfrac{dx}{dt}$ again:

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt}.$$

Method — A worked example

For $x = t^2$, $y = t^3$: $\dfrac{dy}{dx} = \dfrac{3t}{2}$. Then $\dfrac{d}{dt}\left(\dfrac{3t}{2}\right) = \dfrac{3}{2}$, and $\dfrac{dx}{dt} = 2t$, so

$$\frac{d^2y}{dx^2} = \frac{3/2}{2t} = \frac{3}{4t}.$$

Method — Concavity

At $t = 1$, $\dfrac{d^2y}{dx^2} = \frac{3}{4} > 0$: the curve is **concave up** there. A negative value means concave down.

Method — The two-step routine

1. Find $\dfrac{dy}{dx}$ (a function of t). 2. Differentiate it w.r.t. t and divide by $\dfrac{dx}{dt}$.

Practice 2.1 ▪ 1 mark

State the formula for $\dfrac{d^2y}{dx^2}$ for a parametric curve.

Solution 2.1

Worked steps

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}(dy/dx)}{dx/dt} \quad \textbf{B1}.$$

Practice 2.2 ▪ 1 mark

For $x = t^2$, $y = t^3$, given $\dfrac{dy}{dx} = \dfrac{3t}{2}$, find $\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)$.

Solution 2.2

Worked steps

$$\frac{d}{dt}\left(\frac{3t}{2}\right) = \frac{3}{2} \quad \boxed{\textbf{B1}}.$$

Practice 2.3 ▪ 2 marks

Complete: find $\frac{d^2y}{dx^2}$ for that curve.

Solution 2.3

Method: Concavity

Worked steps

$$\frac{d^2y}{dx^2} = \frac{3/2}{2t} = \frac{3}{4t} \quad \boxed{\textbf{M1}}\boxed{\textbf{A1}}.$$

Exam-style 3.1 ▪ 1 mark

The second derivative of a parametric curve is

A $\dfrac{d^2y/dt^2}{d^2x/dt^2}$

B $\dfrac{\frac{d}{dt}(dy/dx)}{dx/dt}$

C $\dfrac{dy/dt}{dx/dt}$

D $\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)$

Solution 3.1

Method: The correct formula

A $\dfrac{d^2y/dt^2}{d^2x/dt^2}$

B $\dfrac{\frac{d}{dt}(dy/dx)}{dx/dt}$ ✓

C $\dfrac{dy/dt}{dx/dt}$

D $\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)$

Worked steps

B — differentiate dy/dx w.r.t. t, then divide by dx/dt

Exam-style 3.2 ▪ 1 mark

A curve has $x = t$, $y = t^3 - 3t$.

(a) Find $\dfrac{dy}{dx}$.

(b) Find $\dfrac{d^2y}{dx^2}$.

(c) State the concavity at $t = 1$.

Solution 3.2

Method: Concavity

Worked steps

(a) $\dfrac{dy}{dx} = \dfrac{3t^2 - 3}{1} = 3t^2 - 3$ **B1**. (b) $\dfrac{d}{dt}(3t^2 - 3) = 6t$; divide by $dx/dt = 1$: $\dfrac{d^2y}{dx^2} = 6t$ **M1** **A1**. (c) at $t = 1$, $6 > 0$: concave up **B1**.

Exam-style 3.3 ▪ 4 marks

For $x = t^2$, $y = t^2 + t$, find $\dfrac{d^2y}{dx^2}$ at $t = 1$.

Solution 3.3

Worked steps

$\dfrac{dy}{dx} = \dfrac{2t+1}{2t}$ **M1**; $\dfrac{d}{dt}\left(\dfrac{2t+1}{2t}\right) = \dfrac{d}{dt}(1 + \frac{1}{2t}) = -\dfrac{1}{2t^2}$ **M1**; divide by $2t$: $\dfrac{d^2y}{dx^2} = -\dfrac{1}{4t^3}$ **M1**; at $t = 1$: $-\frac{1}{4}$ **A1**.