Past-paper walk-through

Volume with Disc Method: Revolving Around the x- or y-Axis ▪ 8.9

eXercise

Questions in this set

- 2026 Q5
- 2022 Q5
- 2021 Q3
- 2016 Q5

Reading the mark scheme

- **B1** a mark that stands on its own – the point is either made or not.
- **M1** a *method* mark: the right approach, even if the number is wrong.
- **A1** an *answer* mark, and it usually needs its M mark first.
- **C1** a working mark on the way to the answer.
- // separates answers the examiner accepts equally.
- **ecf** = error carried forward: a later part is marked on your own earlier answer.
- **owtte** = “or words to that effect” – the wording need not match.

Question 5

2026 ▪ Q5

Let f be the function defined by $f(x) = \sqrt[3]{x-1}$, and let g be the function defined by $g(x) = e^{(-2x+4)}$. The graphs of f and g intersect at the point $(2, 1)$. Let R be the region bounded by the graph of f, the x-axis, and the vertical line $x = 2$, as shown in the figure.

[Graph in the xy-plane: y-axis marked at 1, x-axis marked at 1, 2, 3. The curve $y = f(x)$ starts near $x = 1$ (where it crosses the x-axis) and rises to the right, passing through $(2, 1)$; the curve $y = g(x)$ decreases from upper left, also passing through $(2, 1)$, and continues decreasing toward the right past $x = 3$. The shaded region R is bounded below by the x-axis, on the left by the curve $y = f(x)$, and on the right by the vertical line $x = 2$, lying between $x = 1$ and $x = 2$.]

(a) Evaluate $\int_1^2 f(x)\, dx$. Show the work that leads to your answer.

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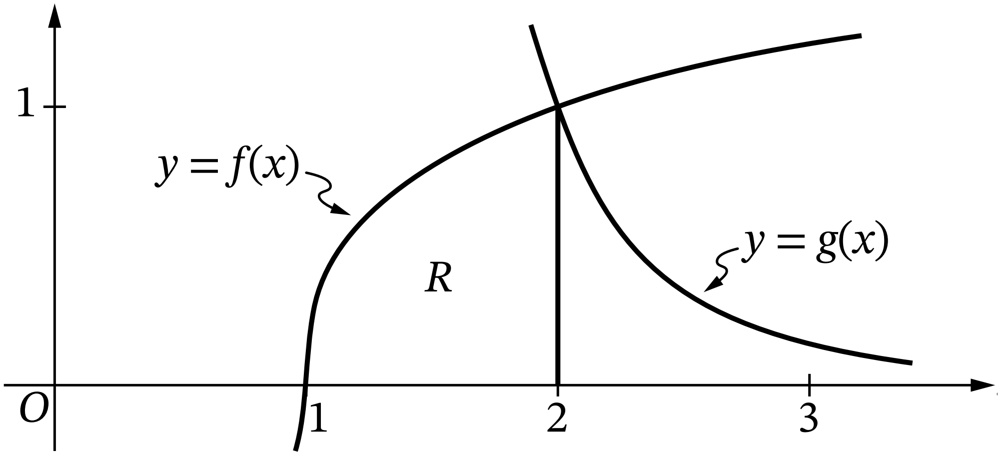
(b) Write, but do not evaluate, an expression involving one or more integrals that gives the volume of the solid generated when region R is rotated about the x-axis.

(c) Write, but do not evaluate, an expression involving one or more integrals that gives the perimeter of region R.

(d) Evaluate $\displaystyle\int_2^{\infty} g(x)\,dx$. Show the work that leads to your answer.

Question 5



Question 5

2022 ▪ Q5

[Figure 1: Graph of $y = \frac{1}{x}$ in the first quadrant; x-axis marked at 1 and 5, curve decreasing from near the y-axis at $x = 1$ down toward the x-axis; the shaded region R is bounded by the curve, the x-axis, and the vertical lines $x = 1$ and $x = 5$, labelled "Figure 1".]

[Figure 2: Graph of $y = \frac{1}{x^2}$ in the first quadrant; x-axis marked at 3, curve decreasing from near the y-axis at $x = 3$ down toward the x-axis extending to the right unbounded; the region W is the unbounded region between the curve and the x-axis to the right of $x = 3$, labelled "Figure 2".]

Figures 1 and 2, shown above, illustrate regions in the first quadrant associated with the graphs of $y = \frac{1}{x}$ and $y = \frac{1}{x^2}$, respectively. In Figure 1, let R be the region bounded

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by the graph of $y = \dfrac{1}{x}$, the x-axis, and the vertical lines $x = 1$ and $x = 5$. In Figure 2, let W be the unbounded region between the graph of $y = \dfrac{1}{x^2}$ and the x-axis that lies to the right of the vertical line $x = 3$.

(a) Find the area of region R.

(b) Region R is the base of a solid. For the solid, at each x the cross section perpendicular to the x-axis is a rectangle with area given by $xe^{x/5}$. Find the volume of the solid.

(c) Find the volume of the solid generated when the unbounded region W is revolved about the x-axis.

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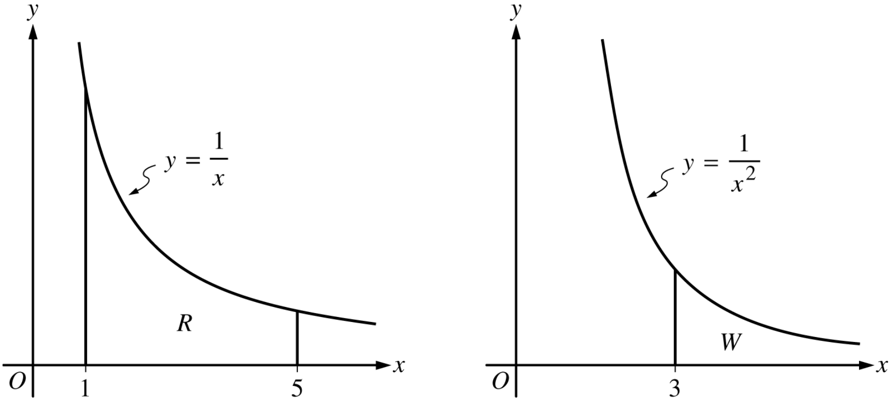


Figure 1 Figure 2

Solution 5

(a)

Mark scheme

- Area $= \int_1^5 \frac{1}{x}\,dx = \ln x\Big|_1^5 = \ln 5 - \ln 1 = \ln 5$. 2 points: 1 for the definite integral with correct limits ($x = 1$ to $x = 5$), 1 for the correct evaluated answer $\ln 5$.

Solution 5

(b) Mark scheme

- Volume $= \int_1^5 xe^{x/5}\,dx$. Integration by parts with $u = x$, $dv = e^{x/5}dx \Rightarrow du = dx$, $v = 5e^{x/5}$: $\int xe^{x/5}dx = 5xe^{x/5} - \int 5e^{x/5}dx = 5xe^{x/5} - 25e^{x/5} + C = 5e^{x/5}(x-5) + C$. Volume $= 5e^{x/5}(x-5)\Big|_1^5 = 5e^0(0) - 5e^{1/5}(-4) = 20e^{1/5}$. 4 points: 1 for the correct definite integral (a nonzero constant multiple is acceptable), 1 for correctly setting up integration by parts (u and dv), 1 for the antiderivative $5xe^{x/5} - \int 5e^{x/5}dx$ (or equivalent, e.g. tabular method), 1 for the final correct answer $20e^{1/5}$.

Solution 5

(c)

Mark scheme

- Volume $= \pi \int_3^\infty \left(\frac{1}{x^2}\right)^2 dx = \pi \lim_{b\to\infty} \int_3^b \frac{1}{x^4}\, dx = \pi \lim_{b\to\infty} \left[-\frac{1}{3x^3}\right]_3^b =$ $\pi \lim_{b\to\infty} \left(-\frac{1}{3}\right) \left[\frac{1}{b^3} - \frac{1}{27}\right] = \pi \left(-\frac{1}{3}\right) \left(0 - \frac{1}{27}\right) = \frac{\pi}{81}$. 3 points: 1 for setting up the improper integral as a limit (a nonzero constant multiple is acceptable), 1 for a correct antiderivative of the form $1/x^n$ ($n \geq 2$ integer), 1 for the final correct answer $\pi/81$ using correct limit notation (no arithmetic with infinity).

Question 3

2021 ▪ Q3

[Graph in the first quadrant: axes labeled x (horizontal) and y (vertical), origin labeled O. A curve starts at the origin, rises to a rounded peak, then descends steeply to meet the x-axis again at a point to the right, resembling the region under $y = cx\sqrt{4-x^2}$ on $0 \le x \le 2$.]

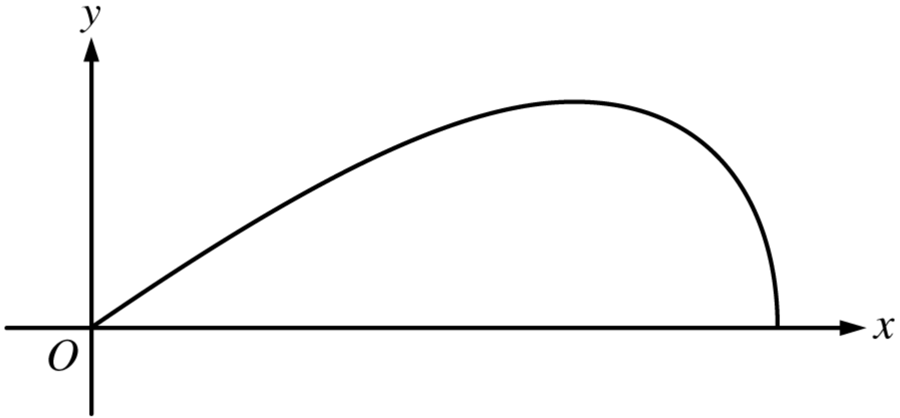
A company designs spinning toys using the family of functions $y = cx\sqrt{4-x^2}$, where c is a positive constant. The figure above shows the region in the first quadrant bounded by the x-axis and the graph of $y = cx\sqrt{4-x^2}$, for some c. Each spinning toy is in the shape of the solid generated when such a region is revolved about the x-axis. Both x and y are measured in inches.

(a) Find the area of the region in the first quadrant bounded by the x-axis and the graph of $y = cx\sqrt{4-x^2}$ for $c = 6$.

Question 3

(b) It is known that, for $y = cx\sqrt{4-x^2}$, $\dfrac{dy}{dx} = \dfrac{c\left(4-2x^2\right)}{\sqrt{4-x^2}}$. For a particular spinning toy, the radius of the largest cross-sectional circular slice is 1.2 inches. What is the value of c for this spinning toy?
(c) For another spinning toy, the volume is 2π cubic inches. What is the value of c for this spinning toy?

Question 3



Solution 3

(a)

Mark scheme

- $6x\sqrt{4-x^2} = 0 \Rightarrow x = 0, 2$. Area $= \displaystyle\int_0^2 6x\sqrt{4-x^2}\,dx$. Let $u = 4 - x^2$: $= 3\int_0^4 u^{1/2}\,du = 2u^{3/2}\Big|_0^4 = 2\cdot 8 = 16$. The area is 16 square inches. (1 pt: correct integrand $cx\sqrt{4-x^2}$ [$c = 6$] in a definite integral, limits need not be correct; 1 pt: correct antiderivative of the form $Ax\sqrt{4-x^2}$-type expression [or equivalent via substitution]; 1 pt: final answer 16.)

Solution 3

(b) Mark scheme

- Set $\dfrac{dy}{dx} = \dfrac{c(4-2x^2)}{\sqrt{4-x^2}} = 0 \Rightarrow x = \sqrt{2}$ is where the largest cross-sectional radius occurs (on $0 < x < 2$). Then $y = c\sqrt{2} \cdot \sqrt{4-2} = 2c$. Since this radius is 1.2: $2c = 1.2 \Rightarrow c = 0.6$. (1 pt: setting $dy/dx = 0$ [or $c(4-2x^2) = 0$]; 1 pt: final answer $c = 0.6$ with supporting work — an unsupported $x = \sqrt{2}$ alone does not earn the first point.)

Solution 3

(c) Mark scheme

- Volume $= \int_0^2 \pi\left(cx\sqrt{4-x^2}\right)^2 dx = \pi c^2 \int_0^2 (4x^2 - x^4)\, dx = \pi c^2 \left[\frac{4}{3}x^3 - \frac{1}{5}x^5\right]_0^2 =$ $\pi c^2\left(\frac{32}{3} - \frac{32}{5}\right) = \dfrac{64\pi c^2}{15}$. Setting equal to 2π: $\dfrac{64\pi c^2}{15} = 2\pi \Rightarrow c^2 = \dfrac{15}{32} \Rightarrow c = \sqrt{\dfrac{15}{32}}$. (1 pt: integrand of the form $A(x\sqrt{4-x^2})^2$ in a definite integral with any nonzero constant A; 1 pt: correct limits $x = 0, 2$ and constant π; 1 pt: correct antiderivative of the presented integrand; 1 pt: final answer $c = \sqrt{15/32}$ — requires the antiderivative point to have been earned.)

Question 5

2016 ▪ Q5

The inside of a funnel of height 10 inches has circular cross sections, as shown in the figure above. At height h, the radius of the funnel is given by $r = \frac{1}{20}\left(3 + h^2\right)$, where $0 \leq h \leq 10$. The units of r and h are inches.

[Diagram of a funnel: a wide cone opening at the top narrowing to a thin stem at the bottom. A horizontal dashed line partway down marks a circular cross section of radius r; a vertical segment labeled h measures the height from that cross section down to the bottom of the funnel's wide part.]

(a) Find the average value of the radius of the funnel.

(b) Find the volume of the funnel.

(c) The funnel contains liquid that is draining from the bottom. At the instant when the height of the liquid is $h = 3$ inches, the radius of the surface of the liquid is

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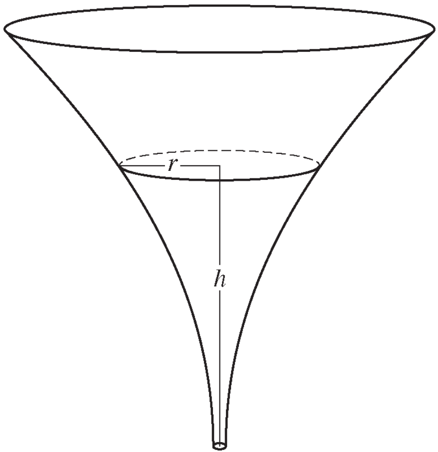
decreasing at a rate of $\frac{1}{5}$ inch per second. At this instant, what is the rate of change of the height of the liquid with respect to time?

Question 5



Solution 5

(a)

Mark scheme

- Average radius $= \dfrac{1}{10}\int_0^{10}\dfrac{1}{20}(3+h^2)\,dh = \dfrac{1}{200}\left[3h+\dfrac{h^3}{3}\right]_0^{10} =$ $\dfrac{1}{200}\left(\left(30+\dfrac{1000}{3}\right)-0\right) = \dfrac{109}{60}$ inches. Award 1 point for the correct averaging integral, 1 point for the correct antiderivative, and 1 point for the answer $109/60$.

Solution 5

(b) Mark scheme

- Volume $= \pi \int_0^{10} \left(\frac{1}{20}(3 + h^2) \right)^2 dh = \frac{\pi}{400} \int_0^{10} (9 + 6h^2 + h^4)\, dh =$ $\frac{\pi}{400} \left[9h + 2h^3 + \frac{h^5}{5} \right]_0^{10} = \frac{\pi}{400} \left(\left(90 + 2000 + \frac{100000}{5} \right) - 0 \right) = \frac{2209\pi}{40}$ cubic inches. Award 1 point for the correct integrand (disk method), 1 point for the correct antiderivative, and 1 point for the answer $2209\pi/40$.

Solution 5

(c) Mark scheme

- $r = \frac{1}{20}(3 + h^2)$, so $\frac{dr}{dt} = \frac{1}{20}(2h)\frac{dh}{dt}$ by the chain rule. At $h = 3$ with $\frac{dr}{dt} = -\frac{1}{5}$: $-\frac{1}{5} = \frac{1}{20}(2 \cdot 3)\frac{dh}{dt} = \frac{3}{10}\frac{dh}{dt}$, so $\frac{dh}{dt} = -\frac{1}{5} \cdot \frac{10}{3} = -\frac{2}{3}$ inches per second. Award 2 points for correctly applying the chain rule to relate dr/dt and dh/dt, and 1 point for the answer $-2/3$ in/sec.