Exercise walk-through

Volumes with Cross Sections: Triangles and Semicircles ▪ 8.8

eXercise

You must be able to

- use $A = \frac{\sqrt{3}}{4}s^2$ for an **equilateral triangle 等边三角形** slice
- use $A = \frac{\pi}{8}s^2$ for a **semicircular 半圆** slice (diameter $= s$)
- substitute the area formula and integrate

Method — Equilateral-triangle cross sections

A slice that is an equilateral triangle with side s has area $A = \frac{\sqrt{3}}{4}s^2$. If $s = f(x) - g(x)$, then

$$V = \int_a^b \frac{\sqrt{3}}{4}\big(f(x) - g(x)\big)^2\,dx.$$

Method — Semicircular cross sections

A semicircle with **diameter** s has radius $\frac{s}{2}$ and area $\frac{1}{2}\pi\left(\frac{s}{2}\right)^2 = \frac{\pi}{8}s^2$. So

$$V = \int_a^b \frac{\pi}{8}\big(f(x) - g(x)\big)^2\,dx.$$

Method — The method is always the same

Whatever the slice's shape, write its **area** in terms of the gap s, then integrate. Only the constant in front changes: 1 (square), $\frac{\sqrt{3}}{4}$ (triangle), $\frac{\pi}{8}$ (semicircle).

Method — A worked triangle-section volume

Base bounded by $y = \sqrt{x}$ and the x-axis on $[0, 4]$, equilateral-triangle sections: $s = \sqrt{x}$, so $V = \int_0^4 \frac{\sqrt{3}}{4} x \, dx = \frac{\sqrt{3}}{4} \cdot 8 = 2\sqrt{3}$.

Practice 2.1 ▪ 1 mark

State the area of an equilateral triangle with side s.

Solution 2.1

Method: Equilateral-triangle cross sections

Worked steps

$A = \frac{\sqrt{3}}{4} s^2$ **B1**.

Practice 2.2 ▪ 1 mark

State the area of a semicircle with diameter s.

Solution 2.2

Method: Semicircular cross sections

Worked steps

$A = \frac{\pi}{8}s^2$ **B1**.

Practice 2.3 ▪ 3 marks

A solid has equilateral-triangle cross sections with side $s = x$ on $[0, 2]$. Find its volume.

Solution 2.3

Method: Equilateral-triangle cross sections

Worked steps

$$V = \int_0^2 \frac{\sqrt{3}}{4} x^2 \, dx = \frac{\sqrt{3}}{4} \left[\frac{x^3}{3} \right]_0^2 = \frac{\sqrt{3}}{4} \cdot \frac{8}{3} = \frac{2\sqrt{3}}{3}$$ M1 M1 A1.

Exam-style 3.1 ▪ 1 mark

A semicircular cross section with diameter s has area

A $\frac{\pi}{2}s^2$

B $\frac{\pi}{4}s^2$

C $\frac{\pi}{8}s^2$

D πs^2

Solution 3.1

Method: Semicircular cross sections

A $\frac{\pi}{2}s^2$

B $\frac{\pi}{4}s^2$

C $\frac{\pi}{8}s^2$ ✓

D πs^2

Worked steps

C — radius $s/2$ gives $\frac{1}{2}\pi(s/2)^2 = \frac{\pi}{8}s^2$.

Exam-style 3.2 ▪ 1 mark

The base of a solid is bounded by $y = 2 - x$ and the axes in the first quadrant. Cross sections perpendicular to the x-axis are equilateral triangles.

(a) Write the side length s as a function of x.

(b) Set up the volume integral.

Solution 3.2

Method: A worked triangle-section volume

Worked steps

(a) $s = 2 - x$ **B1**. (b) $\int_0^2 \frac{\sqrt{3}}{4}(2-x)^2\,dx$ **M1** **A1** *area, limits.*

Exam-style 3.3 ▪ 4 marks

The base is the region between $y = \sqrt{x}$ and the x-axis on $[0, 4]$. Cross sections perpendicular to the x-axis are semicircles with diameter across the region. Set up and evaluate the volume.

Solution 3.3

Method: A worked triangle-section volume

Worked steps

$s = \sqrt{x}$ **M1**; $V = \int_0^4 \frac{\pi}{8}(\sqrt{x})^2\,dx = \frac{\pi}{8}\int_0^4 x\,dx$ **M1** $= \frac{\pi}{8}\left[\frac{x^2}{2}\right]_0^4 = \frac{\pi}{8}\cdot 8$ **M1** $= \pi$ **A1**.