Exercise walk-through

Volumes with Cross Sections: Squares and Rectangles ▪ 8.7

eXercise

You must be able to

- write the **side length** of a slice as the gap between two curves
- integrate the **cross-sectional area 横截面积**: $V = \int_a^b A(x)\,dx$
- handle a square ($A = s^2$) or a rectangle cross section

Method — Volume as an integral of area

A solid whose cross section perpendicular to the x-axis has area $A(x)$ has volume

$$V = \int_a^b A(x)\,dx.$$

"Integrate the cross-sectional area" is the whole method.

Method — Square cross sections

If each cross section is a **square** with side equal to the gap between $y = f(x)$ (top) and $y = g(x)$ (bottom), then $s = f(x) - g(x)$ and

$$V = \int_a^b \big(f(x) - g(x)\big)^2 \, dx.$$

Method — A worked square-section volume

Base bounded by $y = \sqrt{x}$ and the x-axis on $[0, 4]$, square cross sections: $s = \sqrt{x}$, so

$$V = \int_0^4 (\sqrt{x})^2 \, dx = \int_0^4 x \, dx = 8.$$

Method — Rectangle cross sections

A rectangle of height $= k\times$ (base) has area $A = k\,s^2$ if height is proportional to the side; otherwise use the stated width and height. Always write $A(x)$ first, then integrate.

Practice 2.1 ▪ 1 mark

A slice has area $A(x) = x^2$. Write the volume integral over $[0, 3]$.

Solution 2.1

Method: Volume as an integral of area

Worked steps

$$\int_0^3 x^2\,dx \quad \boxed{\textbf{B1}}.$$

Practice 2.2 ▪ 3 marks

A solid has square cross sections with side $s = 2x$ on $[0, 2]$. Find its volume.

Solution 2.2

Worked steps

$A = (2x)^2 = 4x^2$ **M1**; $\int_0^2 4x^2\,dx = \left[\frac{4x^3}{3}\right]_0^2 = \frac{32}{3}$ **M1** **A1**.

Practice 2.3 ▪ 2 marks

Evaluate $\int_0^1 (1 - x)^2 \, dx$.

Solution 2.3

Worked steps

$$\int_0^1 (1-x)^2\,dx = \left[-\frac{(1-x)^3}{3}\right]_0^1 = 0 - \left(-\frac{1}{3}\right) = \frac{1}{3} \quad \text{M1}\ \text{A1}.$$

Exam-style 3.1 ▪ 1 mark

The volume of a solid with cross-sectional area $A(x)$ is

A $\displaystyle\int_a^b \sqrt{A(x)}\,dx$

B $\displaystyle\int_a^b A(x)\,dx$

C $\displaystyle\pi\int_a^b A(x)\,dx$

D $\displaystyle\int_a^b A(x)^2\,dx$

Solution 3.1

Method: Volume as an integral of area

A $\int_a^b \sqrt{A(x)}\, dx$

B $\int_a^b A(x)\, dx$ ✓

C $\pi \int_a^b A(x)\, dx$

D $\int_a^b A(x)^2\, dx$

Worked steps

Exam-style 3.2 ▪ 1 mark

The base of a solid is the region between $y = 4 - x^2$ and the x-axis. Cross sections perpendicular to the x-axis are squares.

(a) Write the side length s as a function of x.

(b) Set up the volume integral (with limits).

Solution 3.2

Method: Volume as an integral of area

Worked steps

(a) $s = 4 - x^2$ **B1**. (b) $\int_{-2}^{2} (4 - x^2)^2 \, dx$ **M1** **A1** *area, limits.*

Exam-style 3.3 ▪ 5 marks

The base of a solid is bounded by $y = x$ and $y = x^2$. Cross sections perpendicular to the x-axis are squares with side equal to the gap between the curves. Set up and evaluate the volume.

Solution 3.3

Method: A worked square-section volume

Worked steps

$s = x - x^2$ on $[0,1]$ **M1**; $V = \int_0^1 (x-x^2)^2\,dx = \int_0^1 (x^2 - 2x^3 + x^4)\,dx$ **M1** $= \left[\frac{x^3}{3} - \frac{x^4}{2} + \frac{x^5}{5}\right]_0^1$ **M1** $= \frac{1}{3} - \frac{1}{2} + \frac{1}{5}$ **M1** $= \frac{1}{30}$ **A1**.