Exercise walk-through

Finding the Area Between Curves That Intersect at More Than Two Points ▪ 8.6

eXercise

You must be able to

- find **all** intersection points
- decide which curve is on top on **each** subinterval
- add the pieces (each set up as top minus bottom)

Method — When top and bottom swap

If two curves cross more than twice, the upper curve **changes** between crossings. You must split the integral at each crossing and use the correct top minus bottom on each piece.

Method — Finding all crossings

For $y = x^3$ and $y = x$: $x^3 = x \Rightarrow x(x^2 - 1) = 0 \Rightarrow x = -1, 0, 1$. Three crossings, so **two** subintervals: $[-1, 0]$ and $[0, 1]$.

Method — Deciding top on each piece

On $(0, 1)$ test $x = 0.5$: $x = 0.5$ vs $x^3 = 0.125$, so $y = x$ is on top. On $(-1, 0)$ test $x = -0.5$: $x^3 = -0.125$ vs $x = -0.5$, so $y = x^3$ is on top.

Method — Adding the pieces

$$\text{Area} = \int_{-1}^{0} (x^3 - x)\, dx + \int_{0}^{1} (x - x^3)\, dx = \tfrac{1}{4} + \tfrac{1}{4} = \tfrac{1}{2}.$$

Using $\int |f - g|$ automatically handles the switch, but you must still split to integrate.

Practice 2.1 ▪ 2 marks

Find all solutions of $x^3 = 4x$.

Solution 2.1

Worked steps

$x^3 - 4x = x(x^2 - 4) = 0 \Rightarrow x = 0, \pm 2$ **M1** **A1**.

Practice 2.2 ▪ 1 mark

On $(0, 2)$, which is greater: $y = 4x$ or $y = x^3$? (test $x = 1$)

Solution 2.2

Worked steps

$4x = 4$ vs $x^3 = 1$ at $x = 1$, so $y = 4x$ is greater B1.

Practice 2.3 ▪ 2 marks

Evaluate $\displaystyle\int_0^2 (4x - x^3)\, dx$.

Solution 2.3

Worked steps

$\left[2x^2 - \frac{x^4}{4}\right]_0^2 = 8 - 4 = 4$ **M1** **A1**.

Exam-style 3.1 ▪ 1 mark

When two curves cross three times, the total enclosed area needs

- **A** one integral
- **B** the integral split into two pieces with the correct top each time
- **C** the average of the curves
- **D** integration in y only

Solution 3.1

Method: When top and bottom swap

A one integral

B the integral split into two pieces with the correct top each time ✓

C the average of the curves

D integration in y only

Worked steps

B — split at the crossings, using the correct upper curve on each piece.

Exam-style 3.2 ▪ 2 marks

Find the total area enclosed between $y = x^3$ and $y = 4x$.
(a) Find all intersection points.
(b) By symmetry or otherwise, evaluate the total enclosed area.

Solution 3.2

Worked steps

(a) $x^3 = 4x \Rightarrow x = -2, 0, 2$ **M1** **A1**. (b) On $(0, 2)$ top is $4x$, area $\int_0^2 (4x - x^3)\,dx = 4$; by symmetry the region on $(-2, 0)$ has equal area 4; total $= 8$ **M1** **M1** **M1** **A1**.

Exam-style 3.3 ▪ 2 marks

The curves $y = x^3 - x$ and $y = 0$ enclose two regions. Explain why $\int_{-1}^{1} (x^3 - x)\, dx = 0$ does **not** give the total area, and describe the correct setup.

Solution 3.3

Method: When top and bottom swap

Worked steps

The integral is 0 because the region below the axis (negative signed area) cancels the region above **M1**; the true area is $\int_{-1}^{0}(x^3-x)\,dx+\left|\int_{0}^{1}(x^3-x)\,dx\right|$, i.e. split at $x=0$ and add the absolute areas **A1**.