Exercise walk-through

Finding the Area Between Curves Expressed as Functions of y ▪ 8.5

eXercise

You must be able to

- integrate **right minus left**: $\int_c^d \left(x_{\text{right}} - x_{\text{left}}\right)\, dy$
- rewrite curves as $x = f(y)$
- choose dy to avoid splitting a region

Method — Right minus left

When a region is bounded left and right by curves, integrate in y:

$$\int_c^d \left(x_{\text{right}}(y) - x_{\text{left}}(y)\right)\, dy,$$

where c and d are the lower and upper y-limits.

Method — Rewriting as x of y

$y = \sqrt{x}$ becomes $x = y^2$; $y = x$ stays $x = y$. For the region between $y = \sqrt{x}$ and $y = x$, in y: right curve $x = y$? Test $y = 0.5$: $y^2 = 0.25$ (left), $y = 0.5$ (right), so right $= y$, left $= y^2$.

Method — Setting up the integral

Crossings of $x = y$ and $x = y^2$: $y = y^2 \Rightarrow y = 0, 1$. Area

$$\int_0^1 (y - y^2)\, dy = \left[\frac{y^2}{2} - \frac{y^3}{3}\right]_0^1 = \frac{1}{6}.$$

Method — Why choose dy

Integrating in y turns a sideways-opening region (like one bounded by $x = y^2$) into a **single** integral, instead of splitting a top-and-bottom setup.

Practice 2.1 ▪ 1 mark

Rewrite $y = \sqrt{x}$ as a function $x = f(y)$.

Solution 2.1

Worked steps

$x = y^2$ (for $y \geq 0$) **B1**.

Practice 2.2 ▪ 1 mark

Find where $x = y$ meets $x = y^2$.

Solution 2.2

Worked steps

$y = y^2 \Rightarrow y = 0, 1$ **B1**.

Practice 2.3 ▪ 2 marks

Evaluate $\int_0^2 (4 - y^2)\, dy$.

Solution 2.3

Worked steps

$\left[4y - \frac{y^3}{3}\right]_0^2 = 8 - \frac{8}{3} = \frac{16}{3}$ **M1** **A1**.

Exam-style 3.1 ▪ 1 mark

To find an area by integrating in y, you integrate

A top minus bottom

B right minus left

C $f(x) - g(x)$

D the average of the curves

Solution 3.1

A top minus bottom

B right minus left ✓

C $f(x) - g(x)$

D the average of the curves

Worked steps

B — integrating in y uses right minus left.

Exam-style 3.2 ▪ 2 marks

A region is bounded by $x = y^2$ and $x = 2y$.

(a) Find the y-limits of integration.

(b) Set up and evaluate the area, integrating in y.

Solution 3.2

Worked steps

(a) $y^2 = 2y \Rightarrow y = 0, 2$ **M1** **A1**. (b) right $= 2y$, left $= y^2$; $\int_0^2 (2y - y^2)\, dy = \left[y^2 - \frac{y^3}{3}\right]_0^2 = 4 - \frac{8}{3} = \frac{4}{3}$ **M1** **M1** **M1** **A1**.

Exam-style 3.3 ▪ 4 marks

Find the area enclosed by $x = y^2 - 1$ and $x = 3$ (integrate in y).

Solution 3.3

Worked steps

$y^2 - 1 = 3 \Rightarrow y = \pm 2$ **M1**; right $= 3$, left $= y^2 - 1$; $\int_{-2}^{2} \big(3 - (y^2 - 1)\big)\, dy = \int_{-2}^{2} (4 - y^2)\, dy$ **M1** $= \left[4y - \frac{y^3}{3}\right]_{-2}^{2} = \frac{32}{3}$ **M1** **A1**.