Exercise walk-through

Finding the Area Between Curves Expressed as Functions of x ▪ 8.4

eXercise

You must be able to

- find the **intersection points** that give the limits of integration
- integrate **top minus bottom**: $\int_a^b \left(f_{\text{top}} - f_{\text{bottom}}\right) dx$
- set up the area correctly even when a curve dips below the other

Method — Top minus bottom

The area between $y = f(x)$ (upper) and $y = g(x)$ (lower) from a to b is

$$\int_a^b \big(f(x) - g(x)\big)\, dx.$$

Method — Finding the limits

Set the curves equal to find where they cross. For $y = x$ and $y = x^2$: $x = x^2 \Rightarrow x = 0, 1$. These are the limits.

Method — Which is on top?

Test a point between the crossings. At $x = 0.5$: the line $y = 0.5$ is above the parabola $y = 0.25$, so $y = x$ is the **top**:

$$\int_0^1 (x - x^2)\, dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$

Method — A worked area

Area between $y = 4 - x^2$ and the x-axis ($y = 0$): crossings at $x = \pm 2$, so $\int_{-2}^{2} (4 - x^2)\,dx = \left[4x - \frac{x^3}{3}\right]_{-2}^{2} = \frac{32}{3}$.

Practice 2.1 ▪ 1 mark

Find where $y = x^2$ meets $y = 4$.

Solution 2.1

Worked steps

$x^2 = 4 \Rightarrow x = \pm 2$ $\boxed{\textbf{B1}}$.

Practice 2.2 ▪ 2 marks

Set up (do not evaluate) the integral for the area between $y = x^2$ and $y = 4$.

Solution 2.2

Worked steps

$\int_{-2}^{2} (4 - x^2)\, dx$ **M1** **A1** *top−bottom, limits.*

Practice 2.3 ▪ 2 marks

Evaluate $\int_0^2 (2x - x^2)\,dx$.

Solution 2.3

Worked steps

$\left[x^2 - \frac{x^3}{3}\right]_0^2 = 4 - \frac{8}{3} = \frac{4}{3}$ **M1** **A1**.

Exam-style 3.1 ▪ 1 mark

The area between $y = f(x)$ (upper) and $y = g(x)$ (lower) on $[a, b]$ is

A $\int_a^b (g - f)\,dx$

B $\int_a^b (f - g)\,dx$

C $\int_a^b f\,dx$

D $\int_a^b (f + g)\,dx$

Solution 3.1

Method: Top minus bottom

A $\displaystyle\int_a^b (g - f)\,dx$

B $\displaystyle\int_a^b (f - g)\,dx$ ✓

C $\displaystyle\int_a^b f\,dx$

D $\displaystyle\int_a^b (f + g)\,dx$

Worked steps

Exam-style 3.2 ▪ 1 mark

Find the area of the region enclosed by $y = x$ and $y = x^2$.
(a) Find the limits of integration.
(b) Evaluate the area.

Solution 3.2

Worked steps

(a) $x = 0, 1$ **B1**. (b) $\int_0^1 (x - x^2)\, dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{6}$ **M1** **M1** **A1**.

Exam-style 3.3 ▪ 5 marks

Find the area of the region enclosed by $y = 6x - x^2$ and $y = x$.

Solution 3.3

Worked steps

Intersect: $6x - x^2 = x \Rightarrow x^2 - 5x = 0 \Rightarrow x = 0, 5$ **M1**; top is $6x - x^2$;
$\int_0^5 \left((6x - x^2) - x\right) dx = \int_0^5 (5x - x^2)\, dx$ **M1** $= \left[\frac{5x^2}{2} - \frac{x^3}{3}\right]_0^5$ **M1** $= \frac{125}{2} - \frac{125}{3}$ **M1** $= \frac{125}{6} \approx 20.8$ **A1**.