Exercise walk-through

The Arc Length of a Smooth, Planar Curve and Distance Traveled ▪ 8.13

eXercise

You must be able to

- apply $L = \int_a^b \sqrt{1 + \left(f'(x)\right)^2}\, dx$
- set up the integral from a given function
- recognise the same idea as **distance travelled** along a path

Method — The arc-length formula

The length of the curve $y = f(x)$ from $x = a$ to $x = b$ is

$$L = \int_a^b \sqrt{1 + \left(f'(x)\right)^2}\, dx.$$

It adds up tiny hypotenuses $\sqrt{dx^2 + dy^2}$ along the curve.

Method — Setting up the integral

For $y = \frac{2}{3}x^{3/2}$: $f'(x) = x^{1/2}$, so $\left(f'\right)^2 = x$, and

$$L = \int_0^3 \sqrt{1+x}\, dx.$$

Method — Evaluating

$$\int_0^3 \sqrt{1+x}\,dx = \left[\tfrac{2}{3}(1+x)^{3/2}\right]_0^3 = \tfrac{2}{3}\big(8-1\big) = \tfrac{14}{3}.$$

Method — Distance along a path

If a particle moves along $y = f(x)$, the **distance travelled** is the same arc-length integral. Many arc-length integrals must be evaluated numerically — set them up correctly even when you cannot integrate by hand.

Practice 2.1 ▪ 1 mark

State the arc-length formula for $y = f(x)$ on $[a, b]$.

Solution 2.1

Method: The arc-length formula

Worked steps

$$L = \int_a^b \sqrt{1 + \left(f'(x)\right)^2}\, dx \quad \boxed{\textbf{B1}}.$$

Practice 2.2 ▪ 2 marks

For $y = \frac{2}{3}x^{3/2}$, find $f'(x)$ and $\left(f'(x)\right)^2$.

Solution 2.2

Worked steps

$f'(x) = x^{1/2}$ **M1**; $\left(f'(x)\right)^2 = x$ **A1**.

Practice 2.3 ▪ 2 marks

Set up (do not evaluate) the arc length of $y = x^2$ from $x = 0$ to $x = 2$.

Solution 2.3

Worked steps

$f'(x) = 2x$, so $L = \int_0^2 \sqrt{1 + 4x^2}\, dx$ **M1** **A1**.

Exam-style 3.1 ▪ 1 mark

The arc length of $y = f(x)$ on $[a, b]$ is

A $\int_a^b f(x)\,dx$

B $\int_a^b \sqrt{1 + \left(f'(x)\right)^2}\,dx$

C $\int_a^b f'(x)\,dx$

D $\pi \int_a^b \left(f(x)\right)^2\,dx$

Solution 3.1

A $\displaystyle\int_a^b f(x)\,dx$

B $\displaystyle\int_a^b \sqrt{1 + \left(f'(x)\right)^2}\,dx$ ✓

C $\displaystyle\int_a^b f'(x)\,dx$

D $\displaystyle\pi \int_a^b \left(f(x)\right)^2\,dx$

Worked steps

B — arc length integrates $\sqrt{1 + (f')^2}$.

Exam-style 3.2 ▪ 2 marks

A curve is $y = \frac{2}{3}x^{3/2}$ for $0 \leq x \leq 3$.
(a) Set up the arc-length integral.
(b) Evaluate it.

Solution 3.2

Worked steps

(a) $f'(x) = x^{1/2}$, so $L = \int_0^3 \sqrt{1+x}\,dx$ M1 A1. (b) $\left[\frac{2}{3}(1+x)^{3/2}\right]_0^3 = \frac{2}{3}(8-1) = \frac{14}{3}$ M1 M1 A1.

Exam-style 3.3 ▪ 3 marks

Set up the integral for the length of $y = \ln(\cos x)$ from $x = 0$ to $x = \frac{\pi}{4}$. (Use $\frac{d}{dx} \ln(\cos x) = -\tan x$ and $1 + \tan^2 x = \sec^2 x$.)

Solution 3.3

Worked steps

$f'(x) = -\tan x$, so $1 + (f')^2 = 1 + \tan^2 x = \sec^2 x$ **M1**; $L = \int_0^{\pi/4} \sqrt{\sec^2 x}\, dx = \int_0^{\pi/4} \sec x \, dx$ **M1** **A1**.