Exercise walk-through

Logistic Models with Differential Equations ▪ 7.9

eXercise

You must be able to

- read the **carrying capacity 环境容纳量** L from $\dfrac{dP}{dt} = kP\left(1 - \dfrac{P}{L}\right)$
- state the long-run limit $\lim_{t\to\infty} P = L$
- find where growth is **fastest** $(P = \frac{L}{2})$

Method — The logistic equation

Real growth slows as resources run out. The logistic model is

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{L}\right),$$

where L is the **carrying capacity** — the population level the system can sustain.

Method — Reading L off the equation

L is the value that makes the bracket zero. From $\dfrac{dP}{dt} = 0.03P\left(1 - \dfrac{P}{800}\right)$, the carrying capacity is $L = 800$.

Method — The long-run limit

Whatever the (positive) start, P approaches L: $\lim_{t\to\infty} P = L$. Here $P \to 800$. If P starts above L it decreases toward L; below L it increases toward L.

Method — Fastest growth at L/2

The growth rate $\frac{dP}{dt}$ is largest at the **inflection point** $P = \frac{L}{2}$ — the middle of the S-shaped curve. Here that is $P = 400$.

Practice 2.1 ▪ 1 mark

For $\dfrac{dP}{dt} = 0.5P\left(1 - \dfrac{P}{2000}\right)$, state the carrying capacity.

Solution 2.1

Method: Reading L off the equation

Worked steps

$L = 2000$ **B1**.

Practice 2.2 ▪ 1 mark

For the same model, state $\lim_{t\to\infty} P$ (given $P(0) = 100$).

Solution 2.2

Method: The long-run limit

Worked steps

2000 **B1**.

Practice 2.3 ▪ 1 mark

At what population is growth fastest for that model?

Solution 2.3

Method: The logistic equation

Worked steps

$P = 1000$ (i.e. $L/2$) **B1**.

Exam-style 3.1 ▪ 1 mark

In $\dfrac{dP}{dt} = kP\left(1 - \dfrac{P}{L}\right)$, growth is fastest when $P =$

A 0

B $\frac{L}{4}$

C $\frac{L}{2}$

D L

Solution 3.1

A 0

B $\frac{L}{4}$

C $\frac{L}{2}$ ✓

D L

Worked steps

C — fastest growth at the inflection point $P = \frac{L}{2}$.

Exam-style 3.2 ▪ 1 mark

A fish population follows $\dfrac{dP}{dt} = 0.2P\left(1 - \dfrac{P}{5000}\right)$, with $P(0) = 500$.

(a) State the carrying capacity.

(b) State $\lim\limits_{t\to\infty} P$.

(c) Find the population at which the population grows fastest.

Solution 3.2

Worked steps

(a) 5000 **B1**. (b) 5000 **B1**. (c) 2500 **B1**.

Exam-style 3.3 ▪ 2 marks

A logistic model has carrying capacity 1200 and $P(0) = 1500$. State, with a reason, whether the population increases or decreases toward the carrying capacity.

Solution 3.3

Method: The logistic equation

Worked steps

Decreases — $P(0) = 1500 > 1200 = L$, so $1 - \frac{P}{L} < 0$ and $\dfrac{dP}{dt} < 0$, pulling P down toward 1200 **M1** **A1**.