Exercise walk-through

Exponential Models with Differential Equations ▪ 7.8

eXercise

You must be able to

- write the solution $y = y_0 e^{kt}$ from the model $\dfrac{dy}{dt} = ky$
- find k and y_0 from given data
- use the model to predict values and **half-life** / doubling situations

Method — The exponential solution

Any quantity whose rate of change is proportional to its size, $\dfrac{dy}{dt} = ky$, grows or decays exponentially:

$$y = y_0 e^{kt},$$

where $y_0 = y(0)$ is the starting amount. $k > 0$ is growth; $k < 0$ is decay.

Method — Finding y0 and k from data

A culture starts at 500 and is 2000 after 3 hours. Then $y_0 = 500$ and $2000 = 500e^{3k}$, so $e^{3k} = 4$, $k = \frac{1}{3}\ln 4 \approx 0.462$. Model: $y = 500e^{0.462t}$.

Method — Decay and half-life

For decay, $k < 0$. A substance with $y = y_0 e^{-0.1t}$ reaches **half** when $e^{-0.1t} = \frac{1}{2}$, so $-0.1t = \ln\frac{1}{2}$, $t = \dfrac{\ln 2}{0.1} \approx 6.93$ (the **half-life**).

Method — Predicting a future value

With $y = 500e^{0.462t}$, the amount at $t = 5$ is $500e^{0.462(5)} = 500e^{2.31} \approx 5030$.

Practice 2.1 ▪ 1 mark

Write the solution of $\frac{dy}{dt} = 0.2y$ with $y(0) = 30$.

Solution 2.1

Method: The exponential solution

Worked steps

$y = 30e^{0.2t}$ **B1**.

Practice 2.2 ▪ 2 marks

A sample decays as $y = 80e^{-0.05t}$. Find the amount at $t = 10$.

Solution 2.2

Method: The exponential solution

Worked steps

$80e^{-0.5} = 80(0.6065) \approx 48.5$ **M1** **A1**.

Practice 2.3 ▪ 3 marks

A population doubles: $y = y_0 e^{kt}$ with $y_0 = 100$ and $y(2) = 200$. Find k.

Solution 2.3

Worked steps

$200 = 100e^{2k}$ **M1**; $e^{2k} = 2$ **M1**; $k = \frac{1}{2}\ln 2 \approx 0.347$ **A1**.

Exam-style 3.1 ▪ 1 mark

The solution of $\dfrac{dy}{dt} = ky$ with $y(0) = y_0$ is

A $y = y_0 + kt$

B $y = y_0 e^{kt}$

C $y = kt^2$

D $y = y_0 k^t$

Solution 3.1

Method: The exponential solution

A $y = y_0 + kt$

B $y = y_0 e^{kt}$ ✓

C $y = kt^2$

D $y = y_0 k^t$

Worked steps

B — $\dfrac{dy}{dt} = ky$ gives $y = y_0 e^{kt}$.

Exam-style 3.2 ▪ 3 marks

A radioactive sample follows $\dfrac{dA}{dt} = kA$, starting at $A(0) = 40$ mg, and 30 mg remains after 5 years.

(a) Find k.

(b) Predict the amount after 12 years.

Solution 3.2

Method: The exponential solution

Worked steps

(a) $30 = 40e^{5k}$ M1; $e^{5k} = 0.75$ M1; $k = \frac{1}{5}\ln 0.75 \approx -0.0575$ A1. (b) $A = 40e^{-0.0575(12)} = 40e^{-0.690} \approx 20.0$ mg M1 A1.

Exam-style 3.3 ▪ 2 marks

A bacteria count grows exponentially from 1000 to 8000 in 6 hours.
(a) Find the growth constant k.
(b) How long, from the start, until the count reaches $16\,000$?

Solution 3.3

Method: The exponential solution

Worked steps

(a) $8000 = 1000e^{6k} \Rightarrow e^{6k} = 8 \Rightarrow k = \frac{1}{6}\ln 8 \approx 0.347$ **M1** **A1**. (b) $16000 = 1000e^{kt} \Rightarrow e^{kt} = 16 \Rightarrow t = \dfrac{\ln 16}{0.347} \approx 8$ h **M1** **M1** **A1**.