Exercise walk-through

Approximating Solutions Using Euler's Method ▪ 7.5

eXercise

You must be able to

- apply the update $y_{\text{new}} = y_{\text{old}} + \dfrac{dy}{dx} \cdot \Delta x$
- carry out two or three steps in a table
- decide whether the estimate is an **over-** or **under-estimate** from concavity

Method — The Euler step

Starting from a known point, take a small step of size Δx and update using the local slope:

$$y_{\text{new}} = y_{\text{old}} + \left.\frac{dy}{dx}\right|_{\text{old}} \cdot \Delta x.$$

Method — A worked two-step

Solve $\dfrac{dy}{dx} = x + y$, $y(0) = 1$, with $\Delta x = 0.1$, to estimate $y(0.2)$.

- Step 1 at $(0, 1)$: slope $= 0 + 1 = 1$; $y \approx 1 + 1(0.1) = 1.1$ at $x = 0.1$.
- Step 2 at $(0.1, 1.1)$: slope $= 0.1 + 1.1 = 1.2$; $y \approx 1.1 + 1.2(0.1) = 1.22$ at $x = 0.2$.

So $y(0.2) \approx 1.22$.

Method — A table helps

Keep columns for x, y, slope $\dfrac{dy}{dx}$, and the next y. Smaller Δx gives a closer estimate.

Method — Over- or under-estimate

If the true solution is **concave up**, the tangent-line steps lie **below** it, so Euler's method **under-estimates**; concave down gives an over-estimate.

Practice 2.1 ▪ 1 mark

State the Euler update formula.

Solution 2.1

Method: The Euler step

Worked steps

$$y_{\text{new}} = y_{\text{old}} + \frac{dy}{dx} \cdot \Delta x \quad \boxed{\textbf{B1}}.$$

Practice 2.2 ▪ 2 marks

For $\dfrac{dy}{dx} = 2x$, $y(1) = 3$, $\Delta x = 0.5$, find y at $x = 1.5$ (one step).

Solution 2.2

Method: A worked two-step

Worked steps

slope at $(1,3) = 2$; $y \approx 3 + 2(0.5) = 4$ **M1** **A1**.

Practice 2.3 ▪ 2 marks

Continue: from $(1.5, y)$ take one more step to estimate $y(2)$.

Solution 2.3

Worked steps

slope at $(1.5, 4) = 3$; $y \approx 4 + 3(0.5) = 5.5$ **M1** **A1**.

Exam-style 3.1 ▪ 1 mark

Euler's method approximates a solution using

A the exact antiderivative

B tangent-line steps along the slope field

C a Riemann sum

D the second derivative

Solution 3.1

Method: Over- or under-estimate

A the exact antiderivative

B tangent-line steps along the slope field ✓

C a Riemann sum

D the second derivative

Worked steps

B — it steps along tangent lines given by the slope field.

Exam-style 3.2 ▪ 4 marks

Given $\frac{dy}{dx} = x - y$ with $y(0) = 2$ and $\Delta x = 0.5$, estimate $y(1)$ using two Euler steps. Show the table.

Solution 3.2

Method: A table helps

Worked steps

Step 1 at $(0,2)$: slope $0-2=-2$; $y \approx 2+(-2)(0.5)=1$ at $x=0.5$ **M1** **A1**. Step 2 at $(0.5,1)$: slope $0.5-1=-0.5$; $y \approx 1+(-0.5)(0.5)=0.75$ at $x=1$ **M1** **A1**.

Exam-style 3.3 ▪ 2 marks

For a solution that is concave up, state whether Euler's method over- or under-estimates the true value, and why.

Solution 3.3

Method: Over- or under-estimate

Worked steps

Under-estimate — for a concave-up curve each tangent-line step lies below the true curve **M1** **A1**.