Exercise walk-through

Modeling Situations with Differential Equations ▪ 7.1

eXercise

You must be able to

- write a **differential equation 微分方程** from a verbal statement about a rate
- recognise “proportional to” as multiplication by a constant k
- identify the quantity, its rate $\frac{dy}{dt}$, and what it depends on

Method — “Rate of change” is a derivative

“The population P changes over time t” means the rate is $\frac{dP}{dt}$. The rest of the sentence tells you what it equals.

Method — “Proportional to”

“Proportional to” means “equal to a constant times”. “The rate of growth is proportional to the population” becomes

$$\frac{dP}{dt} = kP.$$

Method — Proportional to a difference

Newton's law of cooling: an object's temperature T changes at a rate proportional to the difference between T and the room temperature T_r:

$$\frac{dT}{dt} = k(T - T_r).$$

Method — Reading the sign of k

A **growing** quantity has $k > 0$; a **decaying** or cooling one has $k < 0$. The differential equation captures the model before you ever solve it.

Practice 2.1 ▪ 2 marks

Write a differential equation for: "the amount A decreases at a rate proportional to A."

Solution 2.1

Worked steps

$\dfrac{dA}{dt} = kA$ with $k < 0$ (or $\dfrac{dA}{dt} = -kA,\ k > 0$) **M1** **A1** *form, sign/constant.*

Practice 2.2 ▪ 1 mark

A tank drains so that its volume V falls at a constant 3 litres per minute. Write the differential equation.

Solution 2.2

Method: Reading the sign of k

Worked steps

$$\frac{dV}{dt} = -3 \quad \textbf{B1}.$$

Practice 2.3 ▪ 1 mark

Write a differential equation for: “the number of bacteria N grows at a rate proportional to N.”

Solution 2.3

Worked steps

$$\frac{dN}{dt} = kN \quad \boxed{\mathbf{B1}}.$$

Exam-style 3.1 ▪ 1 mark

"The rate of change of y is proportional to y" is written

A $y = kt$

B $\dfrac{dy}{dt} = k$

C $\dfrac{dy}{dt} = ky$

D $\dfrac{dy}{dt} = k + y$

Solution 3.1

A $y = kt$

B $\frac{dy}{dt} = k$

C $\frac{dy}{dt} = ky$ ✓

D $\frac{dy}{dt} = k + y$

Worked steps

C — proportional to y means $\frac{dy}{dt} = ky$.

Exam-style 3.2 ▪ 2 marks

A cup of coffee at temperature T cools in a room at 20°C. Its temperature falls at a rate proportional to how much hotter it is than the room.

(a) Write a differential equation for $\dfrac{dT}{dt}$.

(b) State the sign of the constant of proportionality, with a reason.

Solution 3.2

Method: Proportional to a difference

Worked steps

(a) $\dfrac{dT}{dt} = k(T - 20)$ M1 A1. (b) $k < 0$ — the coffee is cooling, so its temperature decreases B1.

Exam-style 3.3 ▪ 2 marks

A savings account earns interest continuously at 4% per year, so the balance B grows at a rate of $0.04B$ per year, while $\$2000$ per year is also added. Write a differential equation for $\dfrac{dB}{dt}$.

Solution 3.3

Worked steps

$$\frac{dB}{dt} = 0.04B + 2000 \quad \boxed{\textbf{M1}}\boxed{\textbf{A1}}.$$