Exercise walk-through

Riemann Sums, Summation Notation, and Definite Integral Notation ▪ 6.3

eXercise

You must be able to

- write a Riemann sum with **summation notation** $\sum$
- recognise the **definite integral 定积分** as the limit of Riemann sums as $n \to \infty$
- read the meaning of the parts of $\int_a^b f(x)\,dx$

Method — Summation notation for a sum

A right Riemann sum with n equal pieces on $[a, b]$ has width $\Delta x = \dfrac{b-a}{n}$ and sample points $x_k = a + k\,\Delta x$:

$$\sum_{k=1}^{n} f(x_k)\,\Delta x.$$

Method — The definite integral as a limit

Letting the rectangles become infinitely thin gives the exact area:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\,\Delta x.$$

The integral **is** this limit — a sum of infinitely many infinitely thin pieces.

Method — Reading the notation

In $\int_a^b f(x)\,dx$: a and b are the **limits of integration**, $f(x)$ is the **integrand**, and dx marks the variable and the "width" of each piece. The result is a **number** (a signed area), not a function.

Method — From a limit back to an integral

A limit like $\lim_{n\to\infty} \sum_{k=1}^{n} \sqrt{1+\frac{3k}{n}} \cdot \frac{3}{n}$ is a right sum with $\Delta x = \frac{3}{n}$ on $[1,4]$, so it equals $\int_1^4 \sqrt{x}\,dx$.

Practice 2.1 ▪ 1 mark

For $\int_{2}^{10} f\,dx$ estimated with $n = 4$ equal subintervals, state Δx.

Solution 2.1

Worked steps

$$\Delta x = \frac{10-2}{4} = 2 \quad \boxed{\textbf{B1}}.$$

Practice 2.2 ▪ 2 marks

Write, in summation notation, a right Riemann sum for $\int_0^2 f\,dx$ with n equal subintervals.

Solution 2.2

Method: Summation notation for a sum

Worked steps

$\Delta x = \dfrac{2}{n}$, $x_k = \dfrac{2k}{n}$; sum $= \displaystyle\sum_{k=1}^{n} f\left(\tfrac{2k}{n}\right) \cdot \tfrac{2}{n}$ **M1** **A1**.

Practice 2.3 ▪ 2 marks

In $\int_1^5 (3x + 2)\, dx$, name the integrand and the limits of integration.

Solution 2.3

Method: Reading the notation

Worked steps

Integrand $3x + 2$ **B1**; limits of integration 1 and 5 **B1**.

Exam-style 3.1 ▪ 1 mark

The definite integral $\int_a^b f\,dx$ is defined as

A an antiderivative of f

B the limit of Riemann sums as $n \to \infty$

C the slope of f

D the average of $f(a)$ and $f(b)$

Solution 3.1

- **A** an antiderivative of f
- **B** the limit of Riemann sums as $n \to \infty$ ✓
- **C** the slope of f
- **D** the average of $f(a)$ and $f(b)$

Worked steps

B — the definite integral is the limit of Riemann sums.

Exam-style 3.2 ▪ 2 marks

Consider $\lim_{n\to\infty} \sum_{k=1}^{n} \left(2 + \frac{4k}{n}\right)^2 \cdot \frac{4}{n}$.

(a) State the width Δx and the interval $[a, b]$.

(b) Write the limit as a definite integral.

Solution 3.2

Method: From a limit back to an integral

Worked steps

(a) $\Delta x = \dfrac{4}{n}$, so $b - a = 4$; the sample $2 + \frac{4k}{n}$ starts at 2, so $[a, b] = [2, 6]$ **M1** **A1**.

(b) $\displaystyle\int_2^6 x^2\, dx$ **M1** **A1**.

Exam-style 3.3 ▪ 1 mark

A definite integral $\int_0^6 v(t)\,dt$ evaluates to a negative number. Explain what a **negative** value tells you about the signed area, in one sentence.

Solution 3.3

Method: Reading the notation

Worked steps

More signed area lies **below** the axis than above it over $[0, 6]$ **B1**.