Exercise walk-through

Approximating Areas with Riemann Sums ▪ 6.2

eXercise

You must be able to

- compute **left**, **right**, and **midpoint 中点** Riemann sums from a function or a table
- compute a **trapezoidal 梯形** estimate
- decide whether an estimate **over-** or **under-estimates** using increasing/decreasing or concavity

Method — Left and right sums

Estimate $\int_0^6 f\,dx$ with 3 equal subintervals of width $\Delta x = 2$, using the table:

x	**0**	**2**	**4**	**6**
$f(x)$	1	4	9	16

- **Left** sum uses the left endpoints: $2(1+4+9) = 28$.
- **Right** sum uses the right endpoints: $2(4+9+16) = 58$.

Method — Midpoint sum

The **midpoint** sum uses the value at the middle of each subinterval. With $f(x) = x^2$ on $[0, 6]$, $n = 3$, $\Delta x = 2$, midpoints $1, 3, 5$: $2\big(f(1) + f(3) + f(5)\big) = 2(1 + 9 + 25) = 70$.

Method — Trapezoidal estimate

A **trapezoidal** sum averages each pair of endpoint heights: $\Delta x\left[\frac{f_0}{2}+f_1+f_2+\frac{f_3}{2}\right]$. For the table: $2\left[\frac{1}{2}+4+9+\frac{16}{2}\right]=2(21.5)=43$. It equals the average of the left and right sums.

Method — Over- or under-estimate

For an **increasing** function, a left sum **under**-estimates and a right sum **over**-estimates. For a **concave-up** function, the trapezoidal sum **over**-estimates and the midpoint sum **under**-estimates.

Practice 2.1 ▪ 2 marks

Find the **left** Riemann sum for $\displaystyle\int_0^9 g\,dx$ with the three subintervals shown.

Solution 2.1

Method: Left and right sums

Worked steps

$\Delta x = 3$; left $= 3(2+5+6)$ **M1** $= 39$ **A1**.

Practice 2.2 ▪ 2 marks

Find the **right** Riemann sum for the same integral.

Solution 2.2

Worked steps

right $= 3(5 + 6 + 10)$ **M1** $= 63$ **A1**.

Practice 2.3 ▪ 2 marks

Find the **trapezoidal** estimate.

Solution 2.3

Worked steps

$3\left[\frac{2}{2}+5+6+\frac{10}{2}\right]=3(17)=51$ M1 A1. (Equivalently the average of 39 and 63.)

Exam-style 3.1 ▪ 1 mark

For an increasing function, a **left** Riemann sum gives an

- **A** over-estimate
- **B** under-estimate
- **C** exact value
- **D** value that depends on concavity

Solution 3.1

Method: Over- or under-estimate

A over-estimate

B under-estimate ✓

C exact value

D value that depends on concavity

Worked steps

B — for an increasing function the left sum under-estimates.

Exam-style 3.2 ▪ 2 marks

A car's speed $v(t)$ in m/s is recorded:

t **(s)**	**0**	**4**	**8**	**12**
$v(t)$	0	12	20	24

(a) Estimate the distance travelled over $0 \leq t \leq 12$ using a **right** Riemann sum with three subintervals.

(b) Is this an over- or under-estimate of the true distance? Give a reason.

Solution 3.2

Method: Over- or under-estimate

Worked steps

(a) $4(12 + 20 + 24) = 224$ m **M1** **A1**. (b) Over-estimate — v is increasing, so right endpoints are the largest values on each subinterval **B1**.

Exam-style 3.3 ▪ 2 marks

The function f is concave up on $[0, 4]$. A student estimates $\int_0^4 f\,dx$ with a trapezoidal sum. State, with a reason, whether the estimate is too large or too small.

Solution 3.3

Method: Over- or under-estimate

Worked steps

Too large — for a concave-up curve each trapezoid lies **above** the curve, so the trapezoidal sum over-estimates **M1** **A1** *reason, conclusion*.