Exercise walk-through

Evaluating Improper Integrals ▪ 6.13

eXercise

You must be able to

- rewrite an improper integral as a **limit 极限**
- evaluate the limit to decide **convergence 收敛** or **divergence 发散**
- handle a discontinuity inside the interval

Method — Infinite upper limit

Replace ∞ with b and take a limit:

$$\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{b\to\infty} \int_1^{b} x^{-2}\,dx = \lim_{b\to\infty} \Big[-\tfrac{1}{x}\Big]_1^{b} = \lim_{b\to\infty} \Big(-\tfrac{1}{b} + 1\Big) = 1.$$

The limit is finite, so the integral **converges** to 1.

Method — A divergent example

$\int_1^{\infty} \frac{1}{x}\,dx = \lim_{b\to\infty} \left[\ln x\right]_1^b = \lim_{b\to\infty} \ln b = \infty.$ The limit is infinite, so it **diverges**.

Method — Discontinuity at an endpoint

$\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx$ is improper at $x = 0$: $\lim_{a\to 0^+} \int_a^1 x^{-1/2}\,dx = \lim_{a\to 0^+} \left[2\sqrt{x}\right]_a^1 = 2 - 0 = 2.$
Converges.

Method — The rule for 1/x^p on [1, ∞)

$\int_1^\infty \frac{1}{x^p}\,dx$ **converges** when $p > 1$ and **diverges** when $p \leq 1$ — a handy check.

Practice 2.1 ▪ 1 mark

Rewrite $\int_1^\infty \frac{1}{x^3}\, dx$ as a limit.

Solution 2.1

Method: A divergent example

Worked steps

$$\lim_{b\to\infty}\int_1^b x^{-3}\,dx \quad \boxed{\textbf{B1}}.$$

Practice 2.2 ▪ 3 marks

Evaluate $\displaystyle\int_1^\infty \frac{1}{x^3}\,dx$.

Solution 2.2

Worked steps

$\left[-\frac{1}{2x^2}\right]_1^b = -\frac{1}{2b^2} + \frac{1}{2}$ **M1** **M1**; as $b \to \infty$ this $\to \frac{1}{2}$ **A1**.

Practice 2.3 ▪ 1 mark

State whether $\int_1^\infty \frac{1}{x^{1/2}}\, dx$ converges or diverges (use the p-rule).

Solution 2.3

Method: The rule for 1/x^p on [1, ∞)

Worked steps

Diverges — $p = \frac{1}{2} \leq 1$ **B1**.

Exam-style 3.1 ▪ 1 mark

$\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ converges when

A $p < 1$

B $p = 1$

C $p > 1$

D all p

Solution 3.1

Method: The rule for 1/x^p on [1, ∞)

A $p < 1$

B $p = 1$

C $p > 1$ ✓

D all p

Worked steps

C — converges for $p > 1$.

Exam-style 3.2 ▪ 1 mark

Evaluate $\int_0^\infty e^{-x}\,dx$.

(a) Write it as a limit.

(b) Evaluate.

Solution 3.2

Method: A divergent example

Worked steps

(a) $\lim_{b\to\infty}\int_0^b e^{-x}\,dx$ **B1**. (b) $\left[-e^{-x}\right]_0^b = -e^{-b}+1$ **M1** **M1**; $\to 1$ as $b\to\infty$ **A1**.

Exam-style 3.3 ▪ 3 marks

Determine whether $\int_2^\infty \frac{1}{x}\,dx$ converges or diverges, showing your reasoning with a limit.

Solution 3.3

Worked steps

$\lim_{b\to\infty}\left[\ln x\right]_2^b = \lim_{b\to\infty}(\ln b - \ln 2)$ M1 M1; $\ln b \to \infty$, so it **diverges** A1.