

Exercise walk-through

Integrating Using Linear Partial Fractions ■ 6.12

eXercise

You must be able to

- factor the denominator and write the **partial-fraction** 部分分式 decomposition
- solve for the unknown constants
- integrate each simple fraction to a **logarithm**

Method — The decomposition

A proper fraction with a factored denominator splits into simpler pieces:

$$\frac{1}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}.$$

Method — Solving for A and B

Multiply through by $(x - 1)(x + 2)$: $1 = A(x + 2) + B(x - 1)$. Set $x = 1$:
 $1 = 3A \Rightarrow A = \frac{1}{3}$. Set $x = -2$: $1 = -3B \Rightarrow B = -\frac{1}{3}$.

Method — Integrating the pieces

Each piece integrates to a log:

$$\int \left(\frac{1/3}{x-1} - \frac{1/3}{x+2} \right) dx = \frac{1}{3} \ln |x-1| - \frac{1}{3} \ln |x+2| + C.$$

Method — When to divide first

If the fraction is **improper** (numerator degree $\geq$ denominator), do **long division** first, then apply partial fractions to the remainder.

Practice 2.1 ■ 1 mark

Write the partial-fraction form of $\frac{1}{(x-3)(x+1)}$ (with unknowns A, B).

Solution 2.1

Method: When to divide first

Worked steps

$$\frac{A}{x-3} + \frac{B}{x+1} \quad \text{B1}.$$

Practice 2.2 ■ 3 marks

Find A and B for $\frac{5}{(x-2)(x+3)} = \frac{A}{x-2} + \frac{B}{x+3}$.

Solution 2.2

Worked steps

$$5 = A(x + 3) + B(x - 2); x = 2: 5 = 5A \Rightarrow A = 1; x = -3: 5 = -5B \Rightarrow B = -1$$

M1 A1 A1 .

Practice 2.3 ■ 2 marks

Integrate $\int \left(\frac{2}{x-1} + \frac{3}{x+4} \right) dx$.

Solution 2.3

Method: Integrating the pieces

Worked steps

$$2 \ln |x - 1| + 3 \ln |x + 4| + C \quad \text{M1} \quad \text{A1}.$$

Exam-style 3.1 ■ 1 mark

After decomposing, $\int \frac{A}{x-a} dx =$

A $A \ln |x-a| + C$

B $\frac{A}{(x-a)^2} + C$

C $A(x-a) + C$

D $\frac{A}{2}(x-a)^2 + C$

Solution 3.1

A $A \ln |x - a| + C$



B $\frac{A}{(x - a)^2} + C$

C $A(x - a) + C$

D $\frac{A}{2}(x - a)^2 + C$

Worked steps

A — $\int \frac{A}{x-a} dx = A \ln |x - a| + C.$

Exam-style 3.2 ■ 2 marks

Decompose and integrate $\int \frac{1}{(x-1)(x+1)} dx$.

- (a) Find the constants.
- (b) Complete the integration.

Solution 3.2

Worked steps

(a) $1 = A(x+1) + B(x-1)$; $x = 1$: $A = \frac{1}{2}$; $x = -1$: $B = -\frac{1}{2}$ **M1 A1**. (b) $\frac{1}{2} \ln|x-1| - \frac{1}{2} \ln|x+1| + C$ **M1 A1**.

Exam-style 3.3 ■ 5 marks

Find $\int \frac{x+7}{(x-1)(x+3)} dx$.

Solution 3.3

Worked steps

$$x + 7 = A(x + 3) + B(x - 1) \quad \text{M1}; \quad x = 1: 8 = 4A \Rightarrow A = 2; \quad x = -3: 4 = -4B \Rightarrow B = -1 \quad \text{M1 A1}; \quad \int \left(\frac{2}{x-1} - \frac{1}{x+3} \right) dx = 2 \ln |x-1| - \ln |x+3| + C \quad \text{M1 A1}.$$