Past-paper walk-through

Integrating Using Integration by Parts ▪ 6.11

eXercise

Questions in this set

- 2024 Q5
- 2023 Q5

Reading the mark scheme

- **B1** a mark that stands on its own – the point is either made or not.
- **M1** a *method* mark: the right approach, even if the number is wrong.
- **A1** an *answer* mark, and it usually needs its M mark first.
- **C1** a working mark on the way to the answer.
- // separates answers the examiner accepts equally.
- **ecf** = error carried forward: a later part is marked on your own earlier answer.
- **owtte** = "or words to that effect" – the wording need not match.

Question 5

2024 ▪ Q5

x	$\mathbf{0}$	π	2π
$f'(x)$	5	6	0

The function f is twice differentiable for all x with $f(0) = 0$. Values of f', the derivative of f, are given in the table for selected values of x.

Question 5

(a) For $x \geq 0$, the function h is defined by $h(x) = \int_0^x \sqrt{1 + (f'(t))^2}\, dt$. Find the value of $h'(\pi)$. Show the work that leads to your answer.

(b) What information does $\int_0^\pi \sqrt{1 + (f'(x))^2}\, dx$ provide about the graph of f?

(c) Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $f(2\pi)$. Show the computations that lead to your answer.

(d) Find $\int (t + 5) \cos\left(\frac{t}{4}\right)\, dt$. Show the work that leads to your answer.

Question 5

x	0	π	2π
$f'(x)$	5	6	0

Solution 5

(a) Mark scheme

- For x$\geq 0, h(x) = \int_0^x \sqrt{1 + (f'(t))^2}\, dt, so by the Fundamental Theorem of Calculus h'(x) = \sqrt{1 + (f'(x))^2}. Using f'(\pi) = 6 from the given table, h'(\pi) = \sqrt{1 + 6^2} = \sqrt{37}. Points: 1 for the correct FTC setup h'(x) = \sqrt{1 + (f'(x))^2} (or \sqrt{1 + (f'(\pi))^2}), 1 for the correct final answer \sqrt{37}.$

Solution 5

(b)

Mark scheme

- $\int_0^{\pi} \sqrt{1 + (f'(x))^2}\, dxisthearclengthofthegraphoffontheinterval[0, \pi].Points : 1foridentifying'arclength'(oranequivalentphrasesuchas'distancealongthecurve')explicitlyconne$

Solution 5

(c)

Mark scheme

- Using Euler's method with step size $\pi, startingatx = 0 : f(\pi) \approx f(0) + \pi f'(0) = 0 + 5\pi = 5\pi(usingf(0) = 0andf'(0) = 5fromthetable).Thenf(2\pi) \approx f(\pi) + \pi f'(\pi) \approx 5\pi + \pi(6) = 5\pi + 6\pi = 11\pi(usingf'(\pi) = 6fromthetable).Points : 1fordemonstratingtwoEuler's - methodstepsusingthecorrectexpressionfordy/dx(values5and6fromthetable)withatmostoneerro$ $11\pi(bothpointsrequirethecorrecttablevalues5\pi and6\pi tobecombinedcorrectly).$

Solution 5

(d) Mark scheme

- Integration by parts with u=t+5, dv=$\cos(t/4)\,dt, so du = dt, v = 4\sin(t/4) : \int (t+5)\cos\left(\frac{t}{4}\right)\,dt = 4(t+5)\sin\left(\frac{t}{4}\right) - \int 4\sin\left(\frac{t}{4}\right)\,dt = 4(t+5)\sin\left(\frac{t}{4}\right) + 16\cos\left(\frac{t}{4}\right) + C. Points : 1 for correct u and dv (with correct du and dv, possibly implied via the tabular method), 1 for the correct forr$
  $\int v\,du, i.e. 4(t+5)\sin(t/4) - \int 4\sin(t/4)\,dt (or a mathematically equivalent expression), 1 for the correct final answer 4(t+5)\sin(t/4) + 16\cos(t/4) + C (equivalent forms such as 4t\sin(t/4) + 20\sin(t/4) + 16\cos(t/4) + C also earn this point), including the constant of integration.$

Question 5

2023 ▪ Q5

[Graph on the xy-plane; x-axis from O to 3 with gridlines at 1, 2, 3, y-axis with gridlines at 1, 2, 3, 4. Two curves $y = f(x)$ and $y = g(x)$ are shown for $0 \leq x \leq 3$, both starting at $(0, 4)$ and ending near $(3, 2)$, with $y = f(x)$ lying above $y = g(x)$ in between (concave, bulging shape); the region between the two curves is shaded gray.]

The graphs of the functions f and g are shown in the figure for $0 \leq x \leq 3$. It is known that $g(x) = \dfrac{12}{3 + x}$ for $x \geq 0$. The twice-differentiable function f, which is not explicitly given, satisfies $f(3) = 2$ and $\displaystyle\int_0^3 f(x)\,dx = 10$.

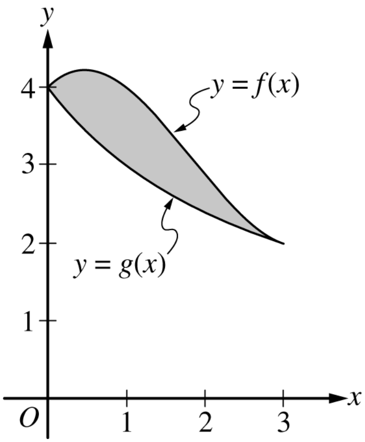
(a) Find the area of the shaded region enclosed by the graphs of f and g.

(b) Evaluate the improper integral $\displaystyle\int_0^\infty (g(x))^2\,dx$, or show that the integral diverges.

Question 5

(c) Let h be the function defined by $h(x) = x \cdot f'(x)$. Find the value of $\int_0^3 h(x)\, dx$.

Question 5



Solution 5

(a)

Mark scheme

- Area $= \int_0^3 (f(x) - g(x))\,dx = \int_0^3 f(x)\,dx - \int_0^3 g(x)\,dx = 10 - \int_0^3 \frac{12}{3+x}\,dx = 10 - 12\big[\ln|3+x|\big]_0^3 = 10 - 12(\ln 6 - \ln 3) = 10 - 12\ln 2$. (1 point for the correct integrand $f(x) - g(x)$, $g(x) - f(x)$, or an absolute-value form, in a definite integral with correct limits — incorrect limits forfeit the third point; 1 point for the correct antiderivative of $g(x)$, of the form $a\ln|3+x|$ or $a\ln(3+x)$; 1 point for the correct final answer $10 - 12\ln 2$, need not be simplified but must be unambiguously communicated.)

Solution 5

(b) Mark scheme

- $\int_0^{\infty}(g(x))^2\,dx = \lim_{b\to\infty}\int_0^b \frac{144}{(3+x)^2}\,dx = \lim_{b\to\infty}\left(-\frac{144}{3+x}\Big|_0^b\right) = \lim_{b\to\infty}\left(-\frac{144}{3+b}+\frac{144}{3}\right) = 48$. The integral converges to 48. (1 point for correct limit notation used throughout, with no arithmetic performed directly with infinity; 1 point for an antiderivative of the form $-\frac{a}{3+x}$ for $a>0$; 1 point for the correct final answer 48, only earned with $a = 144$.)

Solution 5

(c) Mark scheme

- Let $h(x) = x \cdot f'(x)$. Using integration by parts with $u = x$, $dv = f'(x)\,dx$, $du = dx$, $v = f(x)$: $\int h(x)\,dx = \int x \cdot f'(x)\,dx = x \cdot f(x) - \int f(x)\,dx$. Then $\int_0^3 h(x)\,dx = \left[x \cdot f(x)\right]_0^3 - \int_0^3 f(x)\,dx = (3f(3) - 0 \cdot f(0)) - 10 = 3(2) - 0 - 10 = -4$. (1 point for choosing u and dv, or setting up the tabular method with columns beginning x and $f'(x)$; 1 point for $\int h(x)\,dx = x \cdot f(x) - \int f(x)\,dx$; 1 point — only earned if the first two points are earned — for the correct final answer -4.)