Exercise walk-through

Integrating Functions Using Long Division and Completing the Square ▪ 6.10

eXercise

You must be able to

- use **polynomial long division 多项式长除法** when the numerator's degree is $\geq$ the denominator's
- **complete the square 配方** in a denominator to reach an **arctangent** form
- recognise $\displaystyle\int \frac{du}{u^2 + a^2} = \frac{1}{a} \arctan \frac{u}{a} + C$

Method — Long division first

$\int \frac{x^2}{x+1}\,dx$ is top-heavy. Divide: $\frac{x^2}{x+1} = x - 1 + \frac{1}{x+1}$. Then

$$\int \left(x - 1 + \frac{1}{x+1}\right) dx = \frac{x^2}{2} - x + \ln|x+1| + C.$$

Method — A simple 1/x-type

$\int \frac{1}{x-3}\,dx = \ln|x-3| + C$ (substitution $u = x-3$). Watch for these after dividing.

Method — Completing the square

$\int \frac{1}{x^2+4x+5}\,dx$: complete the square, $x^2+4x+5=(x+2)^2+1$. With $u=x+2$,

$$\int \frac{du}{u^2+1} = \arctan(u)+C = \arctan(x+2)+C.$$

Method — The arctangent template

$$\int \frac{1}{u^2 + a^2}\,du = \frac{1}{a}\arctan\frac{u}{a} + C. \quad \text{For} \int \frac{1}{x^2 + 9}\,dx = \tfrac{1}{3}\arctan\tfrac{x}{3} + C.$$

Practice 2.1 ▪ 2 marks

Use long division to rewrite $\dfrac{x+3}{x+1}$ as $1+\dfrac{?}{x+1}$.

Solution 2.1

Method: Long division first

Worked steps

$$\frac{x+3}{x+1} = 1 + \frac{2}{x+1}$$ M1 A1.

Practice 2.2 ▪ 2 marks

Find $\int \frac{1}{x^2 + 16}\, dx$.

Solution 2.2

Worked steps

$\frac{1}{4}\arctan\frac{x}{4}+C$ M1 A1.

Practice 2.3 ▪ 2 marks

Complete the square: write $x^2 + 6x + 13$ in the form $(x + a)^2 + b$.

Solution 2.3

Method: Completing the square

Worked steps

$(x+3)^2+4$ M1 A1.

Exam-style 3.1 ▪ 1 mark

$$\int \frac{1}{x^2 + 25}\, dx =$$

A $\arctan(x) + C$

B $\frac{1}{5}\arctan\frac{x}{5} + C$

C $\ln(x^2 + 25) + C$

D $5\arctan(5x) + C$

Solution 3.1

A $\arctan(x) + C$

B $\frac{1}{5}\arctan\frac{x}{5} + C$ ✓

C $\ln(x^2 + 25) + C$

D $5\arctan(5x) + C$

Worked steps

B — with $a = 5$, $\int \frac{dx}{x^2+25} = \frac{1}{5}\arctan\frac{x}{5} + C$.

Exam-style 3.2 ▪ 3 marks

Find $\int \frac{x+2}{x+1}\,dx$.

Solution 3.2

Worked steps

$\dfrac{x+2}{x+1} = 1 + \dfrac{1}{x+1}$ M1; $\int \left(1 + \frac{1}{x+1}\right) dx = x + \ln|x+1| + C$ M1 A1.

Exam-style 3.3 ▪ 4 marks

Find $\int \frac{1}{x^2+2x+10}\,dx$ by completing the square.

Solution 3.3

Method: Completing the square

Worked steps

$x^2 + 2x + 10 = (x+1)^2 + 9$ M1; with $u = x+1$, $\int \frac{du}{u^2+9}$ M1 $= \frac{1}{3}\arctan\frac{u}{3}$ M1

$= \frac{1}{3}\arctan\frac{x+1}{3} + C$ A1.