Exercise walk-through

Exploring Behaviors of Implicit Relations ▪ 5.12

eXercise

You must be able to

- find where an implicit curve has a **horizontal tangent 水平切线** ($\frac{dy}{dx} = 0$) or a **vertical tangent 垂直切线** ($\frac{dy}{dx}$ undefined)
- decide **concavity 凹凸性** from the sign of $\dfrac{d^2y}{dx^2}$ on the curve
- evaluate these using a point that lies **on** the curve

Method — Horizontal tangents

For $x^2 + y^2 = 25$, implicit differentiation gives $2x + 2y\dfrac{dy}{dx} = 0$, so

$$\frac{dy}{dx} = -\frac{x}{y}.$$

A **horizontal** tangent needs $\dfrac{dy}{dx} = 0$, i.e. the numerator $x = 0$ (with $y \neq 0$): the points $(0, 5)$ and $(0, -5)$.

Method — Vertical tangents

A **vertical** tangent occurs where $\frac{dy}{dx}$ is undefined — the denominator zero, $y = 0$ (with $x \neq 0$): the points $(5, 0)$ and $(-5, 0)$. This matches the circle's shape.

Method — Concavity on an implicit curve

Differentiate $\dfrac{dy}{dx} = -\dfrac{x}{y}$ again (quotient + chain rule):

$$\frac{d^2y}{dx^2} = -\frac{y - x\,\frac{dy}{dx}}{y^2} = -\frac{y - x(-x/y)}{y^2} = -\frac{y^2 + x^2}{y^3} = -\frac{25}{y^3}.$$

Where $y > 0$ this is negative (**concave down**, the top of the circle); where $y < 0$ it is positive (**concave up**).

Practice 2.1 ▪ 1 mark

For $x^2 + y^2 = 25$, state $\dfrac{dy}{dx}$.

Solution 2.1

Worked steps

$$\frac{dy}{dx} = -\frac{x}{y} \quad \boxed{\mathbf{B1}}.$$

Practice 2.2 ▪ 2 marks

Find the point(s) on $x^2 + y^2 = 25$ where the tangent is **horizontal**.

Solution 2.2

Worked steps

Set numerator $x = 0$ **M1**; points $(0, 5)$ and $(0, -5)$ **A1**.

Practice 2.3 ▪ 1 mark

A curve satisfies $\dfrac{dy}{dx} = \dfrac{2x}{y}$. State where its tangent line is **vertical**.

Solution 2.3

Method: Vertical tangents

Worked steps

Vertical where denominator $y = 0$, i.e. at $(\pm 5, 0)$ — where the curve meets the x-axis **B1**.

Exam-style 3.1 ▪ 1 mark

A curve has $\frac{dy}{dx} = \frac{x-1}{y}$. A horizontal tangent occurs where

A $y = 0$

B $x = 0$

C $x = 1$

D $x = y$

Solution 3.1

Method: Vertical tangents

A $y = 0$

B $x = 0$

C $x = 1$ ✓

D $x = y$

Worked steps

C — a horizontal tangent needs the numerator zero, $x - 1 = 0$, so $x = 1$.

Exam-style 3.2 ▪ 2 marks

The curve $x^2 + xy + y^2 = 7$ passes through $(1, 2)$. Its derivative is $\dfrac{dy}{dx} = -\dfrac{2x + y}{x + 2y}$.

(a) Find the slope of the tangent at $(1, 2)$.

(b) Determine whether the tangent at $(1, 2)$ is horizontal, vertical, or neither.

Solution 3.2

Worked steps

(a) $\dfrac{dy}{dx} = -\dfrac{2(1)+2}{1+2(2)} = -\dfrac{4}{5}$ **M1** **A1**. (b) Neither — the slope $-\frac{4}{5}$ is finite and non-zero **B1**.

Exam-style 3.3 ▪ 2 marks

For $x^2 + y^2 = 25$ it can be shown that $\dfrac{d^2y}{dx^2} = -\dfrac{25}{y^3}$. State, with a reason, whether the curve is concave up or concave down at the point $(3, 4)$.

Solution 3.3

Method: Concavity on an implicit curve

Worked steps

$\frac{d^2y}{dx^2} = -\frac{25}{4^3} = -\frac{25}{64} < 0$ **M1**; negative, so the curve is **concave down** at $(3, 4)$ **A1**.