Exercise walk-through

Solving Optimization Problems ▪ 5.11

eXercise

You must be able to

- differentiate the **objective function 目标函数** and solve $f'(x) = 0$ for the **critical points 临界点**
- **justify** that a critical point is a maximum or minimum (sign of f', or the second-derivative test, or the closed-interval candidates)
- give the answer the question asks for, with units

Method — The full optimization method

Maximize the pen area $A(y) = 40y - 2y^2$ on $0 < y < 20$.

1. Differentiate: $A'(y) = 40 - 4y$.
2. Solve $A'(y) = 0$: $y = 10$.
3. Justify: $A''(y) = -4 < 0$, so $y = 10$ gives a **maximum**.
4. Answer: $x = 40 - 2(10) = 20$, so $A = 20 \times 10 = 200\ \mathrm{m}^2$.

Method — Justifying with the first-derivative sign

If the second derivative is awkward, test the sign of f' on each side of the critical point. Here $A'(9) = +4 > 0$ and $A'(11) = -4 < 0$: f rises then falls, so the critical point is a **maximum**. A justification is **required** for full marks.

Method — Minimizing a cost

A can's surface area is $S(r) = 2\pi r^2 + \dfrac{710}{r}$. Then $S'(r) = 4\pi r - \dfrac{710}{r^2}$. Setting $S' = 0$: $4\pi r^3 = 710$, so $r = \left(\dfrac{710}{4\pi}\right)^{1/3} \approx 3.84$ cm. Since $S''(r) = 4\pi + \dfrac{1420}{r^3} > 0$, this is a **minimum**.

Practice 2.1 ▪ 2 marks

For $A(y) = 40y - 2y^2$, find $A'(y)$ and solve $A'(y) = 0$.

Solution 2.1

Worked steps

$A'(y) = 40 - 4y$ **M1**; $40 - 4y = 0 \Rightarrow y = 10$ **A1**.

Practice 2.2 ▪ 3 marks

The area function is $A = 12x - x^2$. Find the width x that maximizes the area, and **justify** it is a maximum.

Solution 2.2

Method: The full optimization method

Worked steps

$A'(x) = 12 - 2x = 0 \Rightarrow x = 6$ **M1** **A1**; $A''(x) = -2 < 0$, so $x = 6$ is a maximum **B1**.

Practice 2.3 ▪ 3 marks

A function $C(x) = x^2 + \frac{16}{x}$ models a cost for $x > 0$. Find the x that minimizes C.

Solution 2.3

Method: Minimizing a cost

Worked steps

$C'(x) = 2x - \dfrac{16}{x^2} = 0$ **M1**; $2x^3 = 16 \Rightarrow x^3 = 8 \Rightarrow x = 2$ **M1** **A1**.

Exam-style 3.1 ▪ 1 mark

A critical point of f has $f'(c) = 0$ and $f''(c) < 0$. The point is a

- **A** local minimum
- **B** local maximum
- **C** point of inflection
- **D** endpoint

Solution 3.1

Method: Justifying with the first-derivative sign

A local minimum

B local maximum ✓

C point of inflection

D endpoint

Worked steps

B — $f'(c) = 0$ with $f''(c) < 0$ is a local maximum.

Exam-style 3.2 ▪ 2 marks

A rectangular field of area $A = x(200 - x)$ m^2 is enclosed, where $0 < x < 200$.
(a) Find the value of x that maximizes A.
(b) Justify that your value gives a maximum.
(c) State the maximum area.

Solution 3.2

Method: The full optimization method

Worked steps

(a) $A = 200x - x^2$, $A'(x) = 200 - 2x = 0 \Rightarrow x = 100$ **M1** **A1**. (b) $A''(x) = -2 < 0$, so it is a maximum **B1**. (c) $A = 100(100) = 10\,000$ m^2 **A1**.

Exam-style 3.3 ▪ 1 mark

An open box is made from card, with volume $V(x) = 150x - \frac{1}{4}x^3$ for $0 < x < \sqrt{600}$.
(a) Show that $V'(x) = 150 - \frac{3}{4}x^2$.
(b) Find the value of x that maximizes the volume, and justify it.

Solution 3.3

Worked steps

(a) $V'(x) = 150 - \frac{3}{4}x^2$ **B1**. (b) $150 - \frac{3}{4}x^2 = 0 \Rightarrow x^2 = 200 \Rightarrow x = \sqrt{200} \approx 14.1$ **M1** **A1**; $V''(x) = -\frac{3}{2}x < 0$ for $x > 0$, so it is a maximum **B1**.