Exercise walk-through

Introduction to Optimization Problems ▪ 5.10

eXercise

You must be able to

- write a quantity to be maximized or minimized as a function of one variable (the **objective function 目标函数**)
- use a **constraint 约束** to eliminate the extra variable
- state the **domain** that makes physical sense for the problem

Method — Naming the objective and the constraint

Optimization always has two ingredients: the **objective** (what you want biggest or smallest) and a **constraint** (a fixed condition). A farmer has 40 m of fence for a rectangular pen against a wall (no fence on the wall side).

- Objective: **area** $A = xy$ (maximize).
- Constraint: fence used $= x + 2y = 40$.

Method — Reducing to one variable

Solve the constraint for one variable and substitute. From $x + 2y = 40$, $x = 40 - 2y$, so

$$A(y) = (40 - 2y)\, y = 40y - 2y^2.$$

Now A depends on the single variable y — ready to differentiate.

Method — Stating a sensible domain

Lengths are positive, so $y > 0$ and $x = 40 - 2y > 0$, giving $0 < y < 20$. The optimization only searches this **domain**; an answer outside it is rejected.

Method — Reading what the question asks for

Check whether the question wants the **variable** (the y that optimizes), the **maximum value** (A itself), or both. A common lost mark is giving y when the question asked for the area.

Practice 2.1 ▪ 2 marks

A rectangle has perimeter 24 cm. Write its **area** as a function of its width x.

Solution 2.1

Method: Naming the objective and the constraint

Worked steps

Perimeter $2x + 2w = 24 \Rightarrow$ length $= 12 - x$ **M1**; $A = x(12 - x) = 12x - x^2$ **A1**.

Practice 2.2 ▪ 2 marks

An open-top box has a square base of side x and volume $32\ \text{cm}^3$. Write its **surface area** (base $+$ 4 sides) as a function of x.

Solution 2.2

Method: Naming the objective and the constraint

Worked steps

Height from $x^2 h = 32$, so $h = 32/x^2$ **M1**; $S = x^2 + 4x\left(\dfrac{32}{x^2}\right) = x^2 + \dfrac{128}{x}$ **A1**.

Practice 2.3 ▪ 1 mark

For the farmer's pen $A(y) = 40y - 2y^2$, state the domain of y.

Solution 2.3

Worked steps

$0 < y < 20$ **B1**.

Exam-style 3.1 ▪ 1 mark

A quantity Q is to be maximized subject to a constraint. What is the **first** step?

A differentiate Q immediately

B write Q as a function of a single variable using the constraint

C set $Q = 0$

D find the second derivative

Solution 3.1

A differentiate Q immediately

B write Q as a function of a single variable using the constraint ✓

C set $Q = 0$

D find the second derivative

Worked steps

B — reduce to a single variable using the constraint before differentiating.

Exam-style 3.2 ▪ 2 marks

A cylindrical can has volume 355 cm^3. Its surface area is $S = 2\pi r^2 + 2\pi rh$.
(a) Use the volume constraint $\pi r^2 h = 355$ to write S as a function of r only.
(b) State the domain of r.

Solution 3.2

Method: Naming the objective and the constraint

Worked steps

(a) $h = \dfrac{355}{\pi r^2}$ **M1**; $S(r) = 2\pi r^2 + 2\pi r\left(\dfrac{355}{\pi r^2}\right) = 2\pi r^2 + \dfrac{710}{r}$ **A1**. (b) $r > 0$ **B1**.

Exam-style 3.3 ▪ 3 marks

A 600 cm^2 sheet of card is folded into an open box with a square base of side x and height h, using all the card $(x^2 + 4xh = 600)$. Write the **volume** V as a function of x alone.

Solution 3.3

Method: Naming the objective and the constraint

Worked steps

$h = \dfrac{600 - x^2}{4x}$ **M1**; $V = x^2 h = x^2\left(\dfrac{600 - x^2}{4x}\right)$ **M1** $= \dfrac{600x - x^3}{4} = 150x - \frac{1}{4}x^3$ **A1**.