Past-paper walk-through

Solving Related Rates Problems ▪ 4.5

eXercise

Questions in this set

- 2025 Q2
- 2022 Q4
- 2019 Q4
- 2018 Q4
- 2018 Q5
- 2017 Q1
- 2016 Q5
- 2014 Q2
- 2014 Q4

Reading the mark scheme

- **B1** a mark that stands on its own – the point is either made or not.
- **M1** a *method* mark: the right approach, even if the number is wrong.
- **A1** an *answer* mark, and it usually needs its M mark first.
- **C1** a working mark on the way to the answer.
- // separates answers the examiner accepts equally.
- **ecf** = error carried forward: a later part is marked on your own earlier answer.
- **owtte** = “or words to that effect” – the wording need not match.

Question 2

2025 ▪ Q2

Curve C is defined by the polar equation $r(\theta) = 2\sin^2\theta$ for $0 \leq \theta \leq \pi$. Curve C and the semicircle $r = \dfrac{1}{2}$ for $0 \leq \theta \leq \pi$ are shown in the xy-plane.

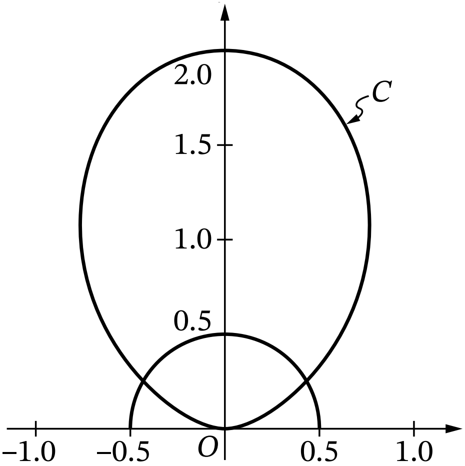
[Graph in the xy-plane, x-axis from -1.0 to 1.0, y-axis from 0 to 2.0 (gridlines at $0.5, 1.0, 1.5, 2.0$). Curve C (labeled with a squiggle and "C") is a large loop-shaped curve passing through the origin O, extending up to about $y = 3$ in shape but shown/gridded to about $y = 2.0$ at its widest visible extent, symmetric about the y-axis, touching the origin and bulging outward to about $x = \pm 1$ near the bottom, reaching its top around $(0, 2)$-ish region. A smaller semicircle of radius $\frac{1}{2}$ (equation $r = \frac{1}{2}$) is nested inside curve C near the origin, spanning from about $(-0.5, 0)$ to $(0.5, 0)$ and bulging up to about $y = 0.5$, overlapping the lower portion of curve C.]

(Note: Your calculator should be in radian mode.)

Question 2

(a) Find the rate of change of *r* with respect to θ at the point on curve *C* where $\theta = 1.3$. Show the setup for your calculations.

Question 2



Question 2

2025 ▪ Q2

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Question 2

(b) Find the area of the region that lies inside curve C but outside the graph of the polar equation $r = \dfrac{1}{2}$. Show the setup for your calculations.

Question 2

2025 ▪ Q2

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Question 2

(c) It can be shown that $\frac{dx}{d\theta} = 4\sin\theta\cos^2\theta - 2\sin^3\theta$ for curve C. For $0 \leq \theta \leq \frac{\pi}{2}$, find the value of θ that corresponds to the point on curve C that is farthest from the y-axis. Justify your answer.

Question 2

2025 ▪ Q2

Curve C is defined by the polar equation $r(\theta) = 2\sin^2\theta$ for $0 \le \theta \le \pi$. Curve C and the semicircle $r = \dfrac{1}{2}$ for $0 \le \theta \le \pi$ are shown in the xy-plane.

[Graph in the xy-plane, x-axis from -1.0 to 1.0, y-axis from 0 to 2.0 (gridlines at $0.5, 1.0, 1.5, 2.0$). Curve C (labeled with a squiggle and "C") is a large loop-shaped curve passing through the origin O, extending up to about $y = 3$ in shape but shown/gridded to about $y = 2.0$ at its widest visible extent, symmetric about the y-axis, touching the origin and bulging outward to about $x = \pm 1$ near the bottom, reaching its top around $(0, 2)$-ish region. A smaller semicircle of radius $\frac{1}{2}$ (equation $r = \frac{1}{2}$) is nested inside curve C near the origin, spanning from about $(-0.5, 0)$ to $(0.5, 0)$ and bulging up to about $y = 0.5$, overlapping the lower portion of curve C.]

(Note: Your calculator should be in radian mode.)

Question 2

(d) A particle travels along curve C so that $\dfrac{d\theta}{dt} = 15$ for all times t. Find the rate at which the particle's distance from the origin changes with respect to time when the particle is at the point where $\theta = 1.3$. Show the setup for your calculations.

Solution 2

(a)

Mark scheme

- $r(\theta) = 2\sin^2\theta \Rightarrow \dfrac{dr}{d\theta} = 4\sin\theta\cos\theta$. At $\theta = 1.3$:
  $\left.\dfrac{dr}{d\theta}\right|_{\theta=1.3} = 4\sin(1.3)\cos(1.3) = 1.031003 \approx 1.031$. (1 pt: answer with setup/differentiation shown, accurate to 3 decimals.)

Solution 2

(b)

Mark scheme

- Curve C meets $r = \frac{1}{2}$ at $\theta_1 = \frac{\pi}{6} = 0.523599$ and $\theta_2 = \frac{5\pi}{6} = 2.617994$. Area $= \dfrac{1}{2} \int_{\theta_1}^{\theta_2} \left[(r(\theta))^2 - \left(\frac{1}{2}\right)^2 \right] d\theta = 2.066769 \approx 2.067$ (or 2.066). (1 pt: integrand including $(r(\theta))^2$; 1 pt: fully correct integrand, e.g. $(r(\theta))^2 - (1/2)^2$; 1 pt: correct final answer including limits and the $\frac{1}{2}$ factor.)

Solution 2

(c)

Mark scheme

- Set $\dfrac{dx}{d\theta} = 4\sin\theta\cos^2\theta - 2\sin^3\theta = 0 \Rightarrow \theta = 0.955317$ (equivalently $\arccos(1/\sqrt{3})$, $\arcsin(\sqrt{2/3})$, or $\arctan(\sqrt{2})$). Candidates test on $[0, \pi/2]$: $x(0) = 0$, $x(0.955317) = 0.769800$, $x(\pi/2) = 0$. The point farthest from the y-axis occurs at $\theta = 0.955$. (1 pt: considers $dx/d\theta = 0$; 1 pt: justification via global/candidates-test argument; 1 pt: correct answer with supporting work.)

Solution 2

(d)

Mark scheme

- $\dfrac{dr}{dt} = \dfrac{dr}{d\theta} \cdot \dfrac{d\theta}{dt}$. At $\theta = 1.3$: $\left.\dfrac{dr}{dt}\right|_{\theta=1.3} = 1.031003 \cdot 15 = 15.465041 \approx 15.465$.
  (1 pt: sets up $dr/d\theta \cdot d\theta/dt$ symbolically or numerically; 1 pt: correct answer.)

Question 4

2022 ▪ Q4

| t (days) | 0 | 3 | 7 | 10 | 12 | |—|—|—|—|—|—| | $r'(t)$ (centimeters per day) | -6.1 | -5.0 | -4.4 | -3.8 | -3.5 |

An ice sculpture melts in such a way that it can be modeled as a cone that maintains a conical shape as it decreases in size. The radius of the base of the cone is given by a twice-differentiable function r, where $r(t)$ is measured in centimeters and t is measured in days. The table above gives selected values of $r'(t)$, the rate of change of the radius, over the time interval $0 \leq t \leq 12$.

(a) Approximate $r''(8.5)$ using the average rate of change of r' over the interval $7 \leq t \leq 10$. Show the computations that lead to your answer, and indicate units of measure.

(b) Is there a time t, $0 \leq t \leq 3$, for which $r'(t) = -6$? Justify your answer.

Question 4

(c) Use a right Riemann sum with the four subintervals indicated in the table to approximate the value of

$$\int_0^{12} r'(t)\,dt.$$

(d) The height of the cone decreases at a rate of 2 centimeters per day. At time $t = 3$ days, the radius is 100 centimeters and the height is 50 centimeters. Find the rate of change of the volume of the cone with respect to time, in cubic centimeters per day, at time $t = 3$ days. (The volume V of a cone with radius r and height h is $V = \frac{1}{3}\pi r^2 h$.)

Question 4

t (days)	0	3	7	10	12
$r'(t)$ (centimeters per day)	-6.1	-5.0	-4.4	-3.8	-3.5

Solution 4

(a)

Mark scheme

- $r''(8.5) \approx \dfrac{r'(10) - r'(7)}{10 - 7} = \dfrac{-3.8 - (-4.4)}{3} = \dfrac{0.6}{3} = 0.2$ centimeters per day per day. 2 points: 1 for the correct difference-quotient setup and value (supporting work must include at least a difference and a quotient), 1 for correct units (cm/day^2).

Solution 4

(b)

Mark scheme

- $r'(0) = -6.1 < -6 < -5.0 = r'(3)$. Since r is twice differentiable, r' is differentiable and therefore continuous on $[0, 3]$. By the Intermediate Value Theorem, there is a time t, $0 < t < 3$, such that $r'(t) = -6$. 2 points: 1 for establishing that -6 lies between $r'(0)$ and $r'(3)$, 1 for the continuity statement plus IVT conclusion that such a t exists.

Solution 4

(c)

Mark scheme

- Right Riemann sum with subintervals $[0,3], [3,7], [7,10], [10,12]$: $\int_0^{12} r'(t)\,dt \approx 3r'(3) + 4r'(7) + 3r'(10) + 2r'(12) = 3(-5.0) + 4(-4.4) + 3(-3.8) + 2(-3.5) = -15 - 17.6 - 11.4 - 7 = -51$. 2 points: 1 for the correct form of the right Riemann sum (at least 7 of the 8 factors correct), 1 for the correct value -51 (units not required).

Solution 4

(d) Mark scheme

- $V = \frac{1}{3}\pi r^2 h \Rightarrow \dfrac{dV}{dt} = \frac{2}{3}\pi r h \dfrac{dr}{dt} + \frac{1}{3}\pi r^2 \dfrac{dh}{dt}$ (product rule on $r^2 h$, chain rule for each differential). At $t = 3$: $r = 100$, $h = 50$, $\dfrac{dr}{dt} = r'(3) = -5$, $\dfrac{dh}{dt} = -2$ (height decreasing 2 cm/day). $\left.\dfrac{dV}{dt}\right|_{t=3} = \frac{2}{3}\pi(100)(50)(-5) + \frac{1}{3}\pi(100)^2(-2) = -\dfrac{70{,}000\pi}{3}$ cubic centimeters per day. 3 points: 1 for the product-rule term, 1 for the chain-rule differentials, 1 for the correct final answer (both prior points required to earn this one).

Question 4

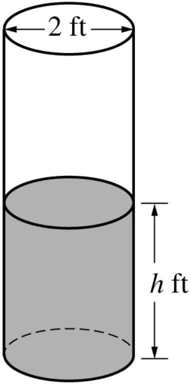
2019 ▪ Q4

[Figure: A vertical cylindrical barrel with diameter labeled "2 ft" across the top. The barrel is shaded (filled with water) from the bottom up to a level marked with a bracket labeled "h ft", with an unshaded empty region above the water level up to the rim.]
A cylindrical barrel with a diameter of 2 feet contains collected rainwater, as shown in the figure above. The water drains out through a valve (not shown) at the bottom of the barrel. The rate of change of the height h of the water in the barrel with respect to time t is modeled by $\frac{dh}{dt} = -\frac{1}{10}\sqrt{h}$, where h is measured in feet and t is measured in seconds. (The volume V of a cylinder with radius r and height h is $V = \pi r^2 h$.)
(a) Find the rate of change of the volume of water in the barrel with respect to time when the height of the water is 4 feet. Indicate units of measure.

Question 4

(b) When the height of the water is 3 feet, is the rate of change of the height of the water with respect to time increasing or decreasing? Explain your reasoning.
(c) At time $t = 0$ seconds, the height of the water is 5 feet. Use separation of variables to find an expression for h in terms of t.

Question 4



Solution 4

(a) Mark scheme

- The barrel is a cylinder of radius 1 ft, so $V = \pi r^2 h = \pi(1)^2 h = \pi h$. Given $\dfrac{dh}{dt} = -\dfrac{1}{10}\sqrt{h}$, $\left.\dfrac{dV}{dt}\right|_{h=4} = \pi \left.\dfrac{dh}{dt}\right|_{h=4} = \pi\left(-\dfrac{1}{10}\sqrt{4}\right) = -\dfrac{\pi}{5}$ cubic feet per second. Points: 1 for $\dfrac{dV}{dt} = \pi\dfrac{dh}{dt}$, 1 for the answer with correct units (cubic feet per second).

Solution 4

(b) Mark scheme

- $\dfrac{d}{dh}\left(-\dfrac{1}{10}\sqrt{h}\right) = -\dfrac{1}{20\sqrt{h}}$. By the chain rule, $\dfrac{d^2h}{dt^2} = -\dfrac{1}{20\sqrt{h}} \cdot \dfrac{dh}{dt} = -\dfrac{1}{20\sqrt{h}} \cdot \left(-\dfrac{1}{10}\sqrt{h}\right) = \dfrac{1}{200}$. Because $\dfrac{d^2h}{dt^2} = \dfrac{1}{200} > 0$ for $h > 0$, the rate of change of the height of the water is increasing when the height of the water is 3 feet. Points: 1 for $\dfrac{d}{dh}\left(-\dfrac{1}{10}\sqrt{h}\right) = -\dfrac{1}{20\sqrt{h}}$, 1 for $\dfrac{d^2h}{dt^2} = -\dfrac{1}{20\sqrt{h}} \cdot \dfrac{dh}{dt}$, 1 for the answer with explanation.

Solution 4

(c) Mark scheme

- Separate variables: $\dfrac{dh}{\sqrt{h}} = -\dfrac{1}{10}\,dt$. Integrate both sides: $\displaystyle\int \frac{dh}{\sqrt{h}} = \int -\frac{1}{10}\,dt \Rightarrow 2\sqrt{h} = -\frac{1}{10}t + C$. Apply the initial condition $h(0) = 5$: $2\sqrt{5} = -\dfrac{1}{10}(0) + C \Rightarrow C = 2\sqrt{5}$. So $2\sqrt{h} = -\dfrac{1}{10}t + 2\sqrt{5}$, giving $h(t) = \left(-\dfrac{1}{20}t + \sqrt{5}\right)^2$. Points: 1 for separation of variables, 1 for antiderivatives, 1 for the constant of integration found using the initial condition, 1 for the final expression $h(t)$. Note: award $0/4$ if no separation of variables is

Solution 4

shown; award at most 2/4 if separation of variables and antiderivatives are correct but no constant of integration is found.

Question 4

2018 ▪ Q4

t **(years)**	**2**	**3**	**5**	**7**	**10**
$H(t)$ (meters)	1.5	2	6	11	15

The height of a tree at time t is given by a twice-differentiable function H, where $H(t)$ is measured in meters and t is measured in years. Selected values of $H(t)$ are given in the table above.

Question 4

(a) Use the data in the table to estimate $H'(6)$. Using correct units, interpret the meaning of $H'(6)$ in the context of the problem.

Question 4

t (years)	2	3	5	7	10
$H(t)$ (meters)	1.5	2	6	11	15