Exercise walk-through

Ratio Test for Convergence ▪ 10.8

eXercise

You must be able to

- compute $L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$
- conclude: $L < 1$ converges, $L > 1$ diverges, $L = 1$ inconclusive
- apply it to series with **factorials** or ***n*th powers**

Method — The ratio test

Compute

$$L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| .$$

- $L < 1$: **converges absolutely**;
- $L > 1$: **diverges**;
- $L = 1$: **inconclusive**.

Method — A factorial example

$$\sum \frac{2^n}{n!}\colon\ \frac{a_{n+1}}{a_n} = \frac{2^{n+1}/(n+1)!}{2^n/n!} = \frac{2}{n+1} \to 0 < 1.$$ **Converges**.

Method — An n-th power example

$$\sum \frac{n}{2^n}\text{:}\ \ \frac{a_{n+1}}{a_n} = \frac{(n+1)/2^{n+1}}{n/2^n} = \frac{n+1}{2n} \to \tfrac{1}{2} < 1.\ \textbf{Converges}.$$

Method — When it is inconclusive

For a p-series, $L = 1$, so the ratio test says nothing — use the p-series rule instead.
The ratio test shines with factorials and exponentials.

Practice 2.1 ▪ 2 marks

For $\sum \frac{3^n}{n!}$, write $\frac{a_{n+1}}{a_n}$.

Solution 2.1

Worked steps

$$\frac{a_{n+1}}{a_n} = \frac{3^{n+1}/(n+1)!}{3^n/n!} = \frac{3}{n+1} \quad \boxed{\textbf{M1}}\boxed{\textbf{A1}}.$$

Practice 2.2 ▪ 2 marks

Find $\lim\limits_{n\to\infty} \dfrac{3}{n+1}$ and state the conclusion.

Solution 2.2

Worked steps

$\dfrac{3}{n+1} \to 0 < 1$, so the series **converges** **M1** **A1**.

Practice 2.3 ▪ 1 mark

For a p-series, the ratio test gives $L = 1$. What does this mean?

Solution 2.3

Method: When it is inconclusive

Worked steps

The ratio test is inconclusive; use another test (the p-series rule) B1.

Exam-style 3.1 ▪ 1 mark

If the ratio test gives $L = 1$, the series

A converges

B diverges

C the test is inconclusive

D equals 1

Solution 3.1

Method: When it is inconclusive

A converges

B diverges

C the test is inconclusive ✓

D equals 1

Worked steps

C — inconclusive when $L = 1$.

Exam-style 3.2 ▪ 3 marks

Apply the ratio test to $\sum_{n=1}^{\infty} \frac{n!}{10^n}$.

(a) Find $\frac{a_{n+1}}{a_n}$ and its limit.

(b) State the conclusion.

Solution 3.2

Method: The ratio test

Worked steps

(a) $\dfrac{a_{n+1}}{a_n} = \dfrac{(n+1)!/10^{n+1}}{n!/10^n} = \dfrac{n+1}{10} \to \infty$ **M1** **M1** **A1**. (b) $L > 1$, so it **diverges** **B1**.

Exam-style 3.3 ▪ 4 marks

Use the ratio test on $\sum \frac{5^n}{n^2}$.

Solution 3.3

Method: The ratio test

Worked steps

$\dfrac{a_{n+1}}{a_n} = \dfrac{5^{n+1}/(n+1)^2}{5^n/n^2} = 5 \cdot \dfrac{n^2}{(n+1)^2} \to 5$ M1 M1 M1; $L = 5 > 1$, so it **diverges** A1.