Exercise walk-through

Comparison Tests for Convergence ▪ 10.6

eXercise

You must be able to

- use the **direct comparison test** with a known series
- use the **limit comparison test** ($\lim a_n/b_n$ finite and positive)
- choose a good comparison series (a p-series or geometric series)

Method — Direct comparison

If $0 \le a_n \le b_n$ and $\sum b_n$ **converges**, then $\sum a_n$ converges. If $a_n \ge b_n \ge 0$ and $\sum b_n$ **diverges**, then $\sum a_n$ diverges.

Method — A direct comparison example

$\sum \dfrac{1}{n^2+1}$: since $\dfrac{1}{n^2+1} \leq \dfrac{1}{n^2}$ and $\sum \frac{1}{n^2}$ converges, the series **converges**.

Method — Limit comparison

If $\lim_{n\to\infty} \frac{a_n}{b_n} = L$ with $0 < L < \infty$, then $\sum a_n$ and $\sum b_n$ do the **same** thing. Easier when the terms only *behave* like a known series.

Method — A limit comparison example

$\sum \dfrac{2n+1}{n^3+5}$: compare with $b_n = \dfrac{1}{n^2}$. $\dfrac{a_n}{b_n} = \dfrac{(2n+1)n^2}{n^3+5} \to 2$, finite and positive. Since $\sum \frac{1}{n^2}$ converges, so does the series.

Practice 2.1 ▪ 1 mark

For $\sum \frac{1}{n^3 + 2}$, name a good comparison series.

Solution 2.1

Worked steps

$\sum \frac{1}{n^3}$ (a convergent p-series, $p = 3$) **B1**.

Practice 2.2 ▪ 2 marks

Is $\dfrac{1}{n^3+2} \leq \dfrac{1}{n^3}$? What does this tell you?

Solution 2.2

Method: A limit comparison example

Worked steps

Yes; since the larger series $\sum \frac{1}{n^3}$ converges, so does $\sum \frac{1}{n^3+2}$ **M1** **A1**.

Practice 2.3 ▪ 2 marks

For limit comparison of $\sum \dfrac{n}{n^2+1}$, a good b_n is $\dfrac{1}{n}$. Find $\lim \dfrac{a_n}{b_n}$.

Solution 2.3

Method: A limit comparison example

Worked steps

$$\frac{a_n}{b_n} = \frac{n/(n^2+1)}{1/n} = \frac{n^2}{n^2+1} \to 1 \quad \text{M1 A1}.$$

Exam-style 3.1 ▪ 1 mark

In the limit comparison test, if $\lim \dfrac{a_n}{b_n} = L$ with $0 < L < \infty$, then

- **A** $\sum a_n$ always converges
- **B** $\sum a_n$ and $\sum b_n$ behave the same way
- **C** $\sum a_n$ always diverges
- **D** the test is inconclusive

Solution 3.1

Method: Limit comparison

A $\sum a_n$ always converges

B $\sum a_n$ and $\sum b_n$ behave the same way ✓

C $\sum a_n$ always diverges

D the test is inconclusive

Worked steps

B — they converge or diverge together.

Exam-style 3.2 ▪ 3 marks

Use direct comparison to show $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n + 1}$ converges.

Solution 3.2

Method: A direct comparison example

Worked steps

$\dfrac{1}{2^n+1} \le \dfrac{1}{2^n}$ **M1**; $\sum \left(\frac{1}{2}\right)^n$ is geometric with $r = \frac{1}{2} < 1$, converges **M1**; so by direct comparison the series converges **A1**.

Exam-style 3.3 ▪ 4 marks

Use limit comparison with $b_n = \frac{1}{n}$ to decide whether $\sum \dfrac{n+2}{n^2+3}$ converges.

Solution 3.3

Method: A limit comparison example

Worked steps

$\frac{a_n}{b_n} = \frac{(n+2)n}{n^2+3} = \frac{n^2+2n}{n^2+3} \to 1$ M1 M1; finite positive, and $\sum \frac{1}{n}$ diverges M1, so the series **diverges** A1.