Exercise walk-through

Harmonic Series and p-Series ▪ 10.5

eXercise

You must be able to

- apply the **p-series** rule: $\sum \frac{1}{n^p}$ converges iff $p > 1$
- recognise the **harmonic series 调和级数** $(p = 1)$ as divergent
- use p-series as the yardstick for comparison tests

Method — The p-series rule

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges if } p > 1, \text{ diverges if } p \leq 1.$$

Method — The harmonic series

The special case $p = 1$, $\sum \frac{1}{n}$, is the **harmonic series** — it **diverges** even though its terms shrink to 0. A must-know fact.

Method — Applying the rule

- $\sum \frac{1}{n^2}$: $p = 2 > 1$, **converges**.
- $\sum \frac{1}{\sqrt{n}} = \sum \frac{1}{n^{1/2}}$: $p = \frac{1}{2} \leq 1$, **diverges**.
- $\sum \frac{1}{n^{3/2}}$: $p = \frac{3}{2} > 1$, **converges**.

Method — The yardstick

p-series are the standard family you compare unknown series to (next subtopic). Know the rule instantly.

Practice 2.1 ▪ 1 mark

State whether $\sum \frac{1}{n^4}$ converges.

Solution 2.1

Method: Applying the rule

Worked steps

Converges ($p = 4 > 1$) **B1**.

Practice 2.2 ▪ 1 mark

State whether $\sum \frac{1}{n^{2/3}}$ converges.

Solution 2.2

Method: Applying the rule

Worked steps

Diverges ($p = \frac{2}{3} \leq 1$) **B1**.

Practice 2.3 ▪ 2 marks

Does the harmonic series converge? Give the value of p.

Solution 2.3

Method: The harmonic series

Worked steps

No, it diverges **B1**; $p = 1$ **B1**.

Exam-style 3.1 ▪ 1 mark

$\sum \frac{1}{n^p}$ converges if and only if

A $p < 1$

B $p = 1$

C $p > 1$

D $p \geq 0$

Solution 3.1

Method: Applying the rule

A $p < 1$

B $p = 1$

C $p > 1$ ✓

D $p \geq 0$

Worked steps

C — converges iff $p > 1$.

Exam-style 3.2 ▪ 1 mark

Classify each as convergent or divergent, stating p:

(a) $\sum \frac{1}{n^{1.2}}$

(b) $\sum \frac{1}{n^{0.9}}$

(c) $\sum \frac{1}{n}$

Solution 3.2

Worked steps

(a) convergent, $p = 1.2 > 1$ **B1**. (b) divergent, $p = 0.9 \leq 1$ **B1**. (c) divergent, $p = 1$ **B1**.

Exam-style 3.3 ▪ 2 marks

Explain why $\sum \frac{1}{\sqrt{n}}$ diverges but $\sum \frac{1}{n^2}$ converges, in terms of p.

Solution 3.3

Method: Applying the rule

Worked steps

$\sum \frac{1}{\sqrt{n}}$ has $p = \frac{1}{2} \leq 1$ (diverges); $\sum \frac{1}{n^2}$ has $p = 2 > 1$ (converges) — the boundary is $p = 1$ **M1** **A1**.