Past-paper walk-through

Integral Test for Convergence ▪ 10.4

eXercise

Questions in this set

- 2021 Q6
- 2017 Q5

Reading the mark scheme

- **B1** a mark that stands on its own – the point is either made or not.
- **M1** a *method* mark: the right approach, even if the number is wrong.
- **A1** an *answer* mark, and it usually needs its M mark first.
- **C1** a working mark on the way to the answer.
- // separates answers the examiner accepts equally.
- **ecf** = error carried forward: a later part is marked on your own earlier answer.
- **owtte** = "or words to that effect" – the wording need not match.

Question 6

2021 ▪ Q6

The function g has derivatives of all orders for all real numbers. The Maclaurin series for g is given by $g(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2e^n + 3}$ on its interval of convergence.

(a) State the conditions necessary to use the integral test to determine convergence of the series $\sum_{n=0}^{\infty} \frac{1}{e^n}$.

Use the integral test to show that $\sum_{n=0}^{\infty} \frac{1}{e^n}$ converges.

Question 6

2021 ▪ Q6

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(b) Use the limit comparison test with the series $\sum_{n=0}^{\infty} \frac{1}{e^n}$ to show that the series $g(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2e^n + 3}$ converges absolutely.

Question 6

2021 ▪ Q6

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(c) Determine the radius of convergence of the Maclaurin series for g.

Question 6

2021 ▪ Q6

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(d) The first two terms of the series $g(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2e^n + 3}$ are used to approximate $g(1)$.

Use the alternating series error bound to determine an upper bound on the error of the approximation.

Solution 6

(a) Mark scheme

- Conditions for the integral test: e^{-x} must be positive, decreasing, and continuous on $[0, \infty)$ — all hold.
  $\int_0^\infty e^{-x}\,dx = \lim_{b\to\infty} \int_0^b e^{-x}\,dx = \lim_{b\to\infty} (-e^{-b} + e^0) = 0 + 1 = 1$, which is finite, so the improper integral converges. Therefore, by the integral test, $\sum_{n=0}^\infty \frac{1}{e^n}$ converges. (1 pt: states all three conditions — positive, decreasing, continuous; 1 pt: correctly writes the improper integral or an equivalent limit; 1 pt: correctly evaluates the limit using proper limit notation, finds e^0 or 1, and concludes convergence — a response that only uses a geometric-series approach earns 0 points here.)

Solution 6

(b) Mark scheme

- $\lim_{n\to\infty} \dfrac{1/e^n}{|(-1)^n/(2e^n+3)|} = \lim_{n\to\infty} \dfrac{2e^n+3}{e^n} = 2$. This limit exists and is positive, so by the limit comparison test (using absolute values), since $\sum_{n=0}^{\infty} \frac{1}{e^n}$ converges, $\sum_{n=0}^{\infty} \left|\frac{(-1)^n}{2e^n+3}\right|$ also converges. Therefore $g(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2e^n+3}$ converges absolutely. (1 pt: correctly sets up the limit comparison with limit notation, with or without absolute values explicit at setup; 1 pt: correctly evaluates the limit as a finite positive number and concludes absolute convergence specifically of $g(1)$ — requires absolute-value reasoning to have appeared.)

Solution 6

(c) Mark scheme

- Ratio test: $\left|\dfrac{a_{n+1}x^{n+1}}{a_n x^n}\right| = \dfrac{2e^n+3}{2e^{n+1}+3}\,|x| \to \dfrac{1}{e}|x|$ as $n \to \infty$. Setting $\dfrac{1}{e}|x| < 1$ gives $|x| < e$. The radius of convergence is $R = e$. (1 pt: correctly sets up the ratio of consecutive terms; 1 pt: correctly computes the limit of the ratio as $|x|/e$ [finite, nonzero coefficient]; 1 pt: explicit final answer $R = e$ from solving the inequality — presenting only an interval like $-e < x < e$ without identifying R does not earn this point.)

Solution 6

(d) Mark scheme

- The terms of the alternating series $g(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2e^n+3}$ decrease in magnitude to 0, so the alternating series error bound applies: the error from using the first two terms ($n = 0, 1$) to approximate $g(1)$ is bounded by the absolute value of the next (third, $n = 2$) term: Error $\leq \left| \frac{(-1)^2}{2e^2+3} \right| = \frac{1}{2e^2+3}$. (1 pt: correct upper bound $\frac{1}{2e^2+3}$.)

Question 5

2017 ▪ Q5

Let f be the function defined by $f(x) = \dfrac{3}{2x^2 - 7x + 5}$.

(a) Find the slope of the line tangent to the graph of f at $x = 3$.

Question 5

2017 ▪ Q5

Let f be the function defined by $f(x) = \dfrac{3}{2x^2 - 7x + 5}$.

(b) Find the x-coordinate of each critical point of f in the interval $1 < x < 2.5$. Classify each critical point as the location of a relative minimum, a relative maximum, or neither. Justify your answers.

Question 5

2017 ▪ Q5

Let f be the function defined by $f(x) = \dfrac{3}{2x^2 - 7x + 5}$.

(c) Using the identity that $\dfrac{3}{2x^2 - 7x + 5} = \dfrac{2}{2x - 5} - \dfrac{1}{x - 1}$, evaluate $\displaystyle\int_5^\infty f(x)\,dx$ or show that the integral diverges.

Question 5

2017 ▪ Q5

Let f be the function defined by $f(x) = \dfrac{3}{2x^2 - 7x + 5}$.

(d) Determine whether the series $\displaystyle\sum_{n=5}^{\infty} \frac{3}{2n^2 - 7n + 5}$ converges or diverges. State the conditions of the test used for determining convergence or divergence.

Solution 5

(a)

Mark scheme

- $f'(x) = \dfrac{-3(4x-7)}{(2x^2-7x+5)^2}$. $f'(3) = \dfrac{(-3)(5)}{(18-21+5)^2} = -\dfrac{15}{4}$. 2 pts for the correct value $f'(3) = -\frac{15}{4}$ (via correct differentiation and substitution).

Solution 5

(b)

Mark scheme

- $f'(x) = 0 \Rightarrow -3(4x - 7) = 0 \Rightarrow x = \frac{7}{4}$. This is the only critical point in $1 < x < 2.5$. f' changes sign from positive to negative at $x = \frac{7}{4}$, so f has a relative maximum at $x = \frac{7}{4}$. 1 pt for the correct x-coordinate $7/4$, 1 pt for correctly classifying it as a relative maximum with justification (sign change of f', e.g. first-derivative test).

Solution 5

(c) Mark scheme

- Using the given partial-fraction identity:
$\int_5^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_5^b \frac{3}{2x^2-7x+5}\,dx = \lim_{b\to\infty} \int_5^b \left(\frac{2}{2x-5} - \frac{1}{x-1}\right) dx = \lim_{b\to\infty} \Big[\ln(2x-5) - \ln(x-1)\Big]_5^b = \lim_{b\to\infty} \ln\left(\frac{2b-5}{b-1}\right) - \ln\left(\frac{5}{4}\right) = \ln 2 - \ln\left(\frac{5}{4}\right) = \ln\left(\frac{8}{5}\right)$. The integral converges to $\ln(8/5)$. 1 pt for the correct antiderivative, 1 pt for the correct limit expression (improper-integral setup), 1 pt for the correct final answer $\ln(8/5)$.

Solution 5

(d) Mark scheme

- f is continuous, positive, and decreasing on $[5, \infty)$, so by the Integral Test, since $\int_5^\infty \dfrac{3}{2x^2 - 7x + 5}\,dx$ converges (from part (c)), the series $\displaystyle\sum_{n=5}^{\infty} \frac{3}{2n^2 - 7n + 5}$ converges. (Alternatively: by the Limit Comparison Test with $\sum 1/n^2$, since terms are positive for $n \geq 5$ and $\lim_{n\to\infty} \dfrac{3/(2n^2 - 7n + 5)}{1/n^2} = \dfrac{3}{2}$ is finite and positive, and $\sum 1/n^2$ converges, so the given series converges.) 2 pts for the correct conclusion (converges) with a fully justified valid test (Integral Test with continuity/positivity/decreasing stated, or Limit Comparison Test with the comparison series and limit correctly stated).

Solution 5