Exercise walk-through

Integral Test for Convergence ▪ 10.4

eXercise

You must be able to

- check the conditions (positive, decreasing, continuous)
- compare $\sum a_n$ with $\int_1^\infty f(x)\,dx$
- conclude convergence or divergence from the integral

Method — The integral test

If $a_n = f(n)$ where f is **positive, decreasing, and continuous** for $x \geq 1$, then $\sum a_n$ and $\int_1^\infty f(x)\,dx$ **both converge or both diverge**.

Method — A worked test

$\sum \dfrac{1}{n^2}$: with $f(x) = \frac{1}{x^2}$, $\displaystyle\int_1^\infty x^{-2}\,dx = 1$ (finite). The integral converges, so the **series converges**.

Method — A divergent case

$\sum \frac{1}{n}$: $\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty} \ln b = \infty$. The integral diverges, so the series diverges.

Method — Check the conditions first

The function must be positive and decreasing (eventually). The integral's **value** is not the series' sum — only its convergence matches.

Practice 2.1 ▪ 2 marks

State the three conditions on f for the integral test.

Solution 2.1

Worked steps

Positive, decreasing, and continuous for $x \geq 1$ **B1** **B1** *any two, all three.*

Practice 2.2 ▪ 2 marks

Evaluate $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$.

Solution 2.2

Worked steps

$\left[-\frac{1}{x}\right]_1^\infty = 0 - (-1) = 1$ **M1** **A1**.

Practice 2.3 ▪ 1 mark

From 2.2, does $\sum \frac{1}{n^2}$ converge?

Solution 2.3

Worked steps

Yes — the integral converges, so the series converges **B1**.

Exam-style 3.1 ▪ 1 mark

The integral test compares a series with

A a geometric series

B an improper integral of the matching function

C the harmonic series

D a partial sum

Solution 3.1

Method: A worked test

A a geometric series

B an improper integral of the matching function ✓

C the harmonic series

D a partial sum

Worked steps

B — with an improper integral of the matching function.

Exam-style 3.2 ▪ 3 marks

Apply the integral test to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$.

(a) Evaluate $\displaystyle\int_1^{\infty} \frac{1}{x^3}\, dx$.

(b) State the conclusion.

Solution 3.2

Method: A worked test

Worked steps

(a) $\int_1^{\infty} x^{-3}\,dx = \left[-\frac{1}{2x^2}\right]_1^{\infty} = 0 + \frac{1}{2} = \frac{1}{2}$ **M1** **M1** **A1**. (b) Converges **B1**.

Exam-style 3.3 ▪ 4 marks

Use the integral test to decide whether $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ converges. (Use the substitution $u = \ln x$.)

Solution 3.3

Method: A worked test

Worked steps

$\displaystyle\int_2^\infty \frac{1}{x\ln x}\,dx$; $u=\ln x$, $du=\frac{1}{x}\,dx$ M1; $=\int \frac{1}{u}\,du=\ln|u|=\ln(\ln x)$ M1; as $x\to\infty$ this $\to\infty$ M1, so it **diverges** A1.