Past-paper walk-through

Representing Functions as Power Series ▪ 10.15

eXercise

Questions in this set

- 2026 Q6
- 2022 Q6
- 2017 Q6
- 2015 Q6

Reading the mark scheme

- **B1** a mark that stands on its own – the point is either made or not.
- **M1** a *method* mark: the right approach, even if the number is wrong.
- **A1** an *answer* mark, and it usually needs its M mark first.
- **C1** a working mark on the way to the answer.
- // separates answers the examiner accepts equally.
- **ecf** = error carried forward: a later part is marked on your own earlier answer.
- **owtte** = "or words to that effect" – the wording need not match.

Question 6

2026 ▪ Q6

The Maclaurin series for a function g is given by

$$\sum_{n=0}^{\infty} 2\left(\frac{-x}{5}\right)^n = 2 - \frac{2}{5}x + \frac{2}{25}x^2 - \frac{2}{125}x^3 + \frac{2}{625}x^4 - \cdots + 2\left(\frac{-x}{5}\right)^n + \cdots$$

and converges to $g(x)$ for $|x| < 5$.

(a) For $x > 0$, the Maclaurin series for g is an alternating geometric series. Find $g(3)$. Show the work that leads to your answer.

Question 6

2026 ▪ Q6

The Maclaurin series for a function g is given by

$$\sum_{n=0}^{\infty} 2\left(\frac{-x}{5}\right)^n = 2 - \frac{2}{5}x + \frac{2}{25}x^2 - \frac{2}{125}x^3 + \frac{2}{625}x^4 - \cdots + 2\left(\frac{-x}{5}\right)^n + \cdots$$

and converges to $g(x)$ for $|x| < 5$.

(b) The function f is defined by $f(x) = g'(x)$. Write the first four nonzero terms of the Maclaurin series for f.

Question 6

2026 ▪ Q6

The Maclaurin series for a function g is given by

$$\sum_{n=0}^{\infty} 2\left(\frac{-x}{5}\right)^n = 2 - \frac{2}{5}x + \frac{2}{25}x^2 - \frac{2}{125}x^3 + \frac{2}{625}x^4 - \cdots + 2\left(\frac{-x}{5}\right)^n + \cdots$$

and converges to $g(x)$ for $|x| < 5$.

(c) The second-degree Taylor polynomial for f about $x = 0$ is used to approximate $f\left(\frac{5}{2}\right)$ as $-\frac{3}{10}$, where f is the function defined in part B. Justify that $\left|f\left(\frac{5}{2}\right) - \left(-\frac{3}{10}\right)\right| \leq \frac{1}{5}$.

Question 6

2026 ▪ Q6

The Maclaurin series for a function g is given by

$$\sum_{n=0}^{\infty} 2\left(\frac{-x}{5}\right)^n = 2 - \frac{2}{5}x + \frac{2}{25}x^2 - \frac{2}{125}x^3 + \frac{2}{625}x^4 - \cdots + 2\left(\frac{-x}{5}\right)^n + \cdots$$

and converges to $g(x)$ for $|x| < 5$.

(d)(i) The function h is defined by $h(x) = 25g(x) - 2e^x$. Write the first three nonzero terms of the Maclaurin series for e^x.

(d)(ii) Write the first three nonzero terms of the Maclaurin series for h.

Question 6

2022 ▪ Q6

The function f is defined by the power series $f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + \frac{(-1)^n x^{2n+1}}{2n+1} + \cdots$ for all real numbers x for which the series converges.

(a) Using the ratio test, find the interval of convergence of the power series for f. Justify your answer.

(b) Show that $\left|f\left(\frac{1}{2}\right) - \frac{1}{2}\right| < \frac{1}{10}$. Justify your answer.

(c) Write the first four nonzero terms and the general term for an infinite series that represents $f'(x)$.

(d) Use the result from part (c) to find the value of $f'\left(\frac{1}{6}\right)$.

Solution 6

(a) Mark scheme

- Ratio test: $\lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{2n+3}/(2n+3)}{(-1)^n x^{2n+1}/(2n+1)}\right| = \lim_{n\to\infty}\left|x^2\frac{2n+1}{2n+3}\right| = |x^2| = x^2$. Need $x^2 < 1 \Rightarrow -1 < x < 1$ (interior of interval). At $x = -1$: series is $-1 + \frac{1}{3} - \frac{1}{5} + \cdots$, alternating with terms decreasing in absolute value to 0, converges by the Alternating Series Test. At $x = 1$: series is $1 - \frac{1}{3} + \frac{1}{5} - \cdots$, also alternating with terms decreasing to 0, converges by AST. So the interval of convergence is $-1 \le x \le 1$. 4 points: 1 for setting up the ratio, 1 for identifying the interior $-1 < x < 1$ from the limit, 1 for considering both endpoints ($x = \pm 1$), 1 for correct analysis at both endpoints and stating the full interval $-1 \le x \le 1$.

Solution 6

(b) Mark scheme

- $f(1/2)$ is an alternating series with first term $\frac{1}{2}$ and terms decreasing in absolute value to 0. By the alternating series error bound, $\left|f(\frac{1}{2}) - \frac{1}{2}\right| < |\text{second term}| = \left|\dfrac{(-1)^1(1/2)^3}{3}\right| = \dfrac{1}{24} < \dfrac{1}{10}$. 2 points: 1 for correctly using $x = \frac{1}{2}$ in the second term of the series, 1 for stating the series is alternating with terms decreasing to 0 and presenting the inequality Error $< \frac{1}{24} < \frac{1}{10}$ (stating Error $= 1/24$ alone does not earn this point).

Solution 6

(c)

Mark scheme

- $f'(x) = 1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n} + \cdots$ (term-by-term derivative of the series for f). 2 points: 1 for presenting the first four nonzero terms $(1, -x^2, x^4, -x^6)$, 1 for the correct general term $(-1)^n x^{2n}$.

Solution 6

(d) Mark scheme

- $f(1/6) = 1 - (1/6)^2 + (1/6)^4 - (1/6)^6 + \cdots$ is a geometric series with first term $a = 1$ and common ratio $r = -1/36$.
  $f(1/6) = \dfrac{a}{1-r} = \dfrac{1}{1-(-1/36)} = \dfrac{1}{37/36} = \dfrac{36}{37}$. 1 point for the correct answer $36/37$, contingent on correctly identifying the series from part (c) as geometric.

Question 6

2017 ▪ Q6

$$f(0) = 0$$

$$f'(0) = 1$$

$$f^{(n+1)}(0) = -n \cdot f^{(n)}(0) \quad \text{for all } n \geq 1$$

A function f has derivatives of all orders for $-1 < x < 1$. The derivatives of f satisfy the conditions above. The Maclaurin series for f converges to $f(x)$ for $|x| < 1$.

(a) Show that the first four nonzero terms of the Maclaurin series for f are $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$, and write the general term of the Maclaurin series for f.

Question 6

2017 ▪ Q6

$$f(0) = 0$$
$$f'(0) = 1$$
$$f^{(n+1)}(0) = -n \cdot f^{(n)}(0) \quad \text{for all } n \geq 1$$

A function f has derivatives of all orders for $-1 < x < 1$. The derivatives of f satisfy the conditions above. The Maclaurin series for f converges to $f(x)$ for $|x| < 1$.

(b) Determine whether the Maclaurin series described in part (a) converges absolutely, converges conditionally, or diverges at $x = 1$. Explain your reasoning.

Question 6

2017 ▪ Q6

$$f(0) = 0$$
$$f'(0) = 1$$
$$f^{(n+1)}(0) = -n \cdot f^{(n)}(0) \quad \text{for all } n \geq 1$$

A function f has derivatives of all orders for $-1 < x < 1$. The derivatives of f satisfy the conditions above. The Maclaurin series for f converges to $f(x)$ for $|x| < 1$.

(c) Write the first four nonzero terms and the general term of the Maclaurin series for $g(x) = \int_0^x f(t)\,dt$.

Question 6

2017 ▪ Q6

$$f(0) = 0$$

$$f'(0) = 1$$

$$f^{(n+1)}(0) = -n \cdot f^{(n)}(0) \quad \text{for all } n \geq 1$$

A function f has derivatives of all orders for $-1 < x < 1$. The derivatives of f satisfy the conditions above. The Maclaurin series for f converges to $f(x)$ for $|x| < 1$.

(d) Let $P_n\left(\frac{1}{2}\right)$ represent the nth-degree Taylor polynomial for g about $x = 0$ evaluated at $x = \frac{1}{2}$, where g is the function defined in part (c). Use the alternating series error bound to show that

Question 6

$$\left| P_4\left(\frac{1}{2}\right) - g\left(\frac{1}{2}\right) \right| < \frac{1}{500}.$$

Solution 6

(a) Mark scheme

- $f(0) = 0$, $f'(0) = 1$, $f''(0) = -1(1) = -1$, $f'''(0) = -2(-1) = 2$, $f^{(4)}(0) = -3(2) = -6$ (from the given recursive derivative relation). The first four nonzero terms of the Maclaurin series: $0 + 1x + \frac{-1}{2!}x^2 + \frac{2}{3!}x^3 + \frac{-6}{4!}x^4 = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$. General term: $\frac{(-1)^{n+1}x^n}{n}$. 1 pt for correctly computing $f''(0)$, $f'''(0)$, $f^{(4)}(0)$, 1 pt for verifying the four given terms match, 1 pt for the correct general term $(-1)^{n+1}x^n/n$.

Solution 6

(b) Mark scheme

- For $x = 1$ the Maclaurin series becomes $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ (the alternating harmonic series). It does not converge absolutely because the harmonic series $\sum 1/n$ diverges. The terms alternate in sign and decrease in magnitude to 0, so by the Alternating Series Test the series converges conditionally. 2 pts for the correct conclusion (converges conditionally) with both parts of the reasoning (harmonic series diverges $\Rightarrow$ not absolute convergence; alternating $+$ decreasing to 0 $\Rightarrow$ converges).

Solution 6

(c) Mark scheme

- $g(x) = \int_0^x f(t)\,dt = \int_0^x \left(t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots + \frac{(-1)^{n+1}t^n}{n} + \cdots\right) dt =$
$\left[\frac{t^2}{2} - \frac{t^3}{3\cdot 2} + \frac{t^4}{4\cdot 3} - \frac{t^5}{5\cdot 4} + \cdots + \frac{(-1)^{n+1}t^{n+1}}{(n+1)n} + \cdots\right]_0^x =$
$\frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{12} - \frac{x^5}{20} + \cdots + \frac{(-1)^{n+1}x^{n+1}}{(n+1)n} + \cdots$. 1 pt for the first two terms $(x^2/2 - x^3/6)$, 1 pt for the remaining shown terms $(x^4/12 - x^5/20)$, 1 pt for the correct general term $(-1)^{n+1}x^{n+1}/[(n+1)n]$.

Solution 6

(d)

Mark scheme

- The terms of the series for $g(1/2)$ alternate in sign and decrease in magnitude to 0, so by the Alternating Series Error Bound, $\left|P_4(\frac{1}{2}) - g(\frac{1}{2})\right|$ is bounded by the magnitude of the first unused (5th) term: $\left|-\dfrac{(1/2)^5}{20}\right|$. Thus $\left|P_4(\frac{1}{2}) - g(\frac{1}{2})\right| \leq \left|-\dfrac{(1/2)^5}{20}\right| = \dfrac{1}{32 \cdot 20} = \dfrac{1}{640} < \dfrac{1}{500}$. 1 pt for correctly applying the alternating series error bound to establish the inequality.

Question 6

2015 ▪ Q6

The Maclaurin series for a function f is given by $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{n} x^n = x - \frac{3}{2}x^2 + 3x^3 - \cdots + \frac{(-3)^{n-1}}{n} x^n + \cdots$ and converges to $f(x)$ for $|x| < R$, where R is the radius of convergence of the Maclaurin series.

(a) Use the ratio test to find R.

(b) Write the first four nonzero terms of the Maclaurin series for f', the derivative of f. Express f' as a rational function for $|x| < R$.

(c) Write the first four nonzero terms of the Maclaurin series for e^x. Use the Maclaurin series for e^x to write the third-degree Taylor polynomial for $g(x) = e^x f(x)$ about $x = 0$.

Solution 6

(a) Mark scheme

- Let a_n be the nth term of the Maclaurin series. $\frac{a_{n+1}}{a_n} = \frac{(-3)^n x^{n+1}}{n+1} \cdot \frac{n}{(-3)^{n-1} x^n} = \frac{-3n}{n+1} x$. $\lim_{n\to\infty} \left| \frac{-3n}{n+1} x \right| = 3|x|$. $3|x| < 1 \Rightarrow |x| < \frac{1}{3}$. So the radius of convergence is $R = \frac{1}{3}$. Award 1 point for setting up the ratio $|a_{n+1}/a_n|$, 1 point for correctly computing the limit of the ratio $(3|x|)$, and 1 point for determining $R = 1/3$.

Solution 6

(b)

Mark scheme

- The first four nonzero terms of the Maclaurin series for f' are $1 - 3x + 9x^2 - 27x^3$ (term-by-term differentiation of the given series for f). As a rational function, $f'(x) = \frac{1}{1-(-3x)} = \frac{1}{1+3x}$ for $|x| < R$. Award 2 points for the correct first four nonzero terms, and 1 point for the correct rational function expression.

Solution 6

(c) Mark scheme

- The first four nonzero terms of the Maclaurin series for e^x are $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$. Multiplying by the Maclaurin series for f: $\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\left(x - \frac{3}{2}x^2 + 3x^3 - \cdots\right) = x - \frac{1}{2}x^2 + 2x^3 + \cdots$. So the third-degree Taylor polynomial for $g(x) = e^x f(x)$ about $x = 0$ is $T_3(x) = x - \frac{1}{2}x^2 + 2x^3$. Award 1 point for the correct first four nonzero terms of the Maclaurin series for e^x, and 2 points for the correct third-degree Taylor polynomial for g.