Exercise walk-through

Representing Functions as Power Series ▪ 10.15

eXercise

You must be able to

- **differentiate** or **integrate** a known power series term by term
- build a new series from the geometric series
- use a series to approximate an integral

Method — Term-by-term calculus

Within its radius of convergence, a power series can be **differentiated** and **integrated** term by term to get the series of the derivative or antiderivative.

Method — Building a new series by integrating

Start from $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$. Integrating gives

$$-\ln(1-x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots .$$

Method — Building by substitution then integrating

$\frac{1}{1+x^2} = \sum (-1)^n x^{2n}$. Integrating term by term gives the series for $\arctan x$:

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots .$$

Method — Approximating an integral

A function like e^{-x^2} has no elementary antiderivative, but its **series** integrates term by term, letting you approximate $\int_0^1 e^{-x^2}\,dx$.

Practice 2.1 ▪ 2 marks

Differentiate the series $\sum_{n=0}^{\infty} x^n$ term by term.

Solution 2.1

Worked steps

$$\frac{d}{dx}\sum_{n=0}^{\infty} x^n = \sum_{n=1}^{\infty} n x^{n-1} \quad \boxed{\textbf{M1}}\boxed{\textbf{A1}}.$$

Practice 2.2 ▪ 2 marks

Write the Maclaurin series for $\dfrac{1}{1-x}$ and integrate its first three terms.

Solution 2.2

Worked steps

$\sum x^n = 1 + x + x^2 + \cdots$; integrating: $x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots$ **M1** **A1**.

Practice 2.3 ▪ 1 mark

State the first three nonzero terms of the series for $\arctan x$.

Solution 2.3

Method: Building by substitution then integrating

Worked steps

$$x - \frac{x^3}{3} + \frac{x^5}{5} \quad \boxed{\textbf{B1}}.$$

Exam-style 3.1 ▪ 1 mark

Within its radius of convergence, a power series may be

A differentiated but not integrated

B integrated but not differentiated

C both differentiated and integrated term by term

D neither

Solution 3.1

Method: Term-by-term calculus

A differentiated but not integrated

B integrated but not differentiated

C both differentiated and integrated term by term ✓

D neither

Worked steps

C — both, term by term, within the radius.

Exam-style 3.2 ▪ 2 marks

Start from $e^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

(a) Write the series for e^{-x^2}.

(b) Write the first three terms of $\displaystyle\int_0^1 e^{-x^2}\,dx$ obtained by integrating that series term by term.

Solution 3.2

Method: Building a new series by integrating

Worked steps

(a) $e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum \frac{(-1)^n x^{2n}}{n!} = 1 - x^2 + \frac{x^4}{2} - \cdots$ **M1** **A1**. (b) $\int_0^1 \left(1 - x^2 + \frac{x^4}{2} - \cdots\right) dx = \left[x - \frac{x^3}{3} + \frac{x^5}{10} - \cdots\right]_0^1 = 1 - \frac{1}{3} + \frac{1}{10} - \cdots$ **M1** **M1** **A1**.

Exam-style 3.3 ▪ 3 marks

Use the geometric series to write a power series for $\dfrac{1}{1-2x}$, and state its radius of convergence.

Solution 3.3

Method: Term-by-term calculus

Worked steps

$$\frac{1}{1-2x} = \sum_{n=0}^{\infty} (2x)^n = \sum_{n=0}^{\infty} 2^n x^n$$ M1 A1; converges for $|2x| < 1$, i.e. $R = \frac{1}{2}$ A1.