Past-paper walk-through

Radius and Interval of Convergence of Power Series ▪ 10.13

eXercise

Questions in this set

- 2025 Q6
- 2024 Q6
- 2022 Q6
- 2021 Q6
- 2018 Q6
- 2016 Q6
- 2015 Q6
- 2014 Q6

Reading the mark scheme

- **B1** a mark that stands on its own – the point is either made or not.
- **M1** a *method* mark: the right approach, even if the number is wrong.
- **A1** an *answer* mark, and it usually needs its M mark first.
- **C1** a working mark on the way to the answer.
- // separates answers the examiner accepts equally.
- **ecf** = error carried forward: a later part is marked on your own earlier answer.
- **owtte** = "or words to that effect" – the wording need not match.

Question 6

2025 ▪ Q6

The Taylor series for a function f about $x = 4$ is given by

$$\sum_{n=1}^{\infty} \frac{(x-4)^{n+1}}{(n+1)3^n} = \frac{(x-4)^2}{2 \cdot 3} + \frac{(x-4)^3}{3 \cdot 3^2} + \frac{(x-4)^4}{4 \cdot 3^3} + \cdots + \frac{(x-4)^{n+1}}{(n+1)3^n} + \cdots$$

and converges to $f(x)$ on its interval of convergence.

(a) Using the ratio test, find the interval of convergence of the Taylor series for f about $x = 4$. Justify your answer.

(b) Find the first three nonzero terms and the general term of the Taylor series for f', the derivative of f, about $x = 4$.

(c) The Taylor series for f' described in part B is a geometric series. For all x in the interval of convergence of the Taylor series for f', show that $f'(x) = \dfrac{x-4}{7-x}$.

Question 6

(d) It is known that the radius of convergence of the Taylor series for f about $x = 4$ is the same as the radius of convergence of the Taylor series for f' about $x = 4$. Does the Taylor series for f' described in part B converge to $f'(x) = \dfrac{x-4}{7-x}$ at $x = 8$? Give a reason for your answer.

Solution 6

(a) Mark scheme

- Ratio test: $\left|\dfrac{(x-4)^{n+2}}{(n+2)3^{n+1}} \cdot \dfrac{(n+1)3^{n}}{(x-4)^{n+1}}\right| = \left|\dfrac{(x-4)(n+1)}{3(n+2)}\right|$. Limit as $n \to \infty$: $\left|\dfrac{x-4}{3}\right|$. This is < 1 when $|x-4| < 3$, i.e. interior $1 < x < 7$. At $x = 1$: series is $\sum \dfrac{(-1)^{n+1} \cdot 3}{n+1}$, converges by the alternating series test. At $x = 7$: series is $\sum \dfrac{3}{n+1}$, diverges by limit comparison to the harmonic series. Interval of convergence: $1 \leq x < 7$. (1 pt: sets up the ratio; 1 pt: correct limit of the ratio, $|x-4|/3$; 1 pt: correct interior of interval, $1 < x < 7$; 1 pt: considers both

Solution 6

endpoints $x = 1$ and $x = 7$; 1 pt: correct analysis at both endpoints and correct final interval $1 \leq x < 7$.)

Solution 6

(b)

Mark scheme

- Differentiating the given series term by term: $f'(x) = \frac{x-4}{3} + \frac{(x-4)^2}{9} + \frac{(x-4)^3}{27} + \cdots + \frac{(x-4)^n}{3^n} + \cdots$. (1 pt: correct first three nonzero terms; 1 pt: correct general term, $(x-4)^n/3^n$.)

Solution 6

(c) Mark scheme

- The series for f is geometric with first term $\dfrac{x-4}{3}$ and common ratio $\dfrac{x-4}{3}$, so $f(x) = \dfrac{\frac{x-4}{3}}{1-\frac{x-4}{3}} = \dfrac{x-4}{3-(x-4)} = \dfrac{x-4}{7-x}$. (1 pt: verification — identifies first term/ratio of the geometric series and simplifies to the given closed form.)

Solution 6

(d)

Mark scheme

- The interior of the interval of convergence found in part A is $1 < x < 7$, and (given) the Taylor series for f' has the same radius of convergence. Since $x = 8$ is outside $1 < x < 7$, the Taylor series for f' does NOT converge to $f'(x) = (x - 4)/(7 - x)$ at $x = 8$. (1 pt: correct 'no' answer with a reason based on $x = 8$ lying outside the interval of convergence.)

Question 6

2024 ▪ Q6

The Maclaurin series for a function f is given by $\sum_{n=1}^{\infty} \frac{(n+1)x^n}{n^2 6^n}$ and converges to $f(x)$ for all x in the interval of convergence. It can be shown that the Maclaurin series for f has a radius of convergence of 6.

(a) Determine whether the Maclaurin series for f converges or diverges at $x = 6$. Give a reason for your answer.

Question 6

2024 ▪ Q6

The Maclaurin series for a function f is given by $\sum_{n=1}^{\infty} \frac{(n+1)x^n}{n^2 6^n}$ and converges to $f(x)$ for all x in the interval of convergence. It can be shown that the Maclaurin series for f has a radius of convergence of 6.

(b) It can be shown that $f(-3) = \sum_{n=1}^{\infty} \frac{(n+1)(-3)^n}{n^2 6^n} = \sum_{n=1}^{\infty} \frac{n+1}{n^2} \left(-\frac{1}{2}\right)^n$ and that the first three terms of this series sum to $S_3 = -\frac{125}{144}$. Show that $|f(-3) - S_3| < \frac{1}{50}$.

Question 6

2024 ▪ Q6

The Maclaurin series for a function f is given by $\sum_{n=1}^{\infty} \frac{(n+1)x^n}{n^2 6^n}$ and converges to $f(x)$ for all x in the interval of convergence. It can be shown that the Maclaurin series for f has a radius of convergence of 6.

(c) Find the general term of the Maclaurin series for f', the derivative of f. Find the radius of convergence of the Maclaurin series for f'.

Question 6

2024 ▪ Q6

The Maclaurin series for a function f is given by $\sum_{n=1}^{\infty} \frac{(n+1)x^n}{n^2 6^n}$ and converges to $f(x)$ for all x in the interval of convergence. It can be shown that the Maclaurin series for f has a radius of convergence of 6.

(d) Let $g(x) = \sum_{n=1}^{\infty} \frac{(n+1)x^{2n}}{n^2 3^n}$. Use the ratio test to determine the radius of convergence of the Maclaurin series for g.

Solution 6

(a) Mark scheme

- The Maclaurin series for f is $\sum_{n=1}^{\infty} \frac{(n+1)x^n}{n^2 6^n}. At x = 6, this becomes \sum_{n=1}^{\infty} \frac{(n+1)6^n}{n^2 6^n} = \sum_{n=1}^{\infty} \frac{n+1}{n^2}. Because \frac{n+1}{n^2} > \frac{1}{n} for all n \geq 1 and the harmonic series \sum \frac{1}{n} diverges, the series \sum \frac{n+1}{n^2} diverges by the comparison test (equivalently$ $1). So the Maclaurin series for f diverges at x = 6. Points: 1 for considering the term (n+1)6^n/(n^2 6^n) (or its simplified form (n+1)/n^2) at x = 6, 1 for a correct comparison-test or limit-comparison-test argument (terms exceed 1/n, or the limit of the ratio to 1/n equals 1) correctly concluding divergence$

Solution 6

(b) Mark scheme

- f(- 3)=$\sum_{n=1}^{\infty} \frac{n+1}{n^2}\left(-\frac{1}{2}\right)^n$ isanalternatingserieswithtermsdecreasinginmagnitudeto0, and$S_3 = -\frac{125}{144}$isthesumofitsfirstthreeterms.Bythealternatingserieserrorbound, usingthefourthterm$(n = 4) : \left|\frac{4+1}{4^2}\left(-\frac{1}{2}\right)^4\right| = \frac{5}{256} < \frac{5}{250} = \frac{1}{50}$.Thus$|f(-3) - S_3| < \frac{1}{50}$.Points : 1forcorrectlyusing$x = -3$inthefourthtermoftheseries(i.e.theexpression$(4+1)/4^2 \cdot (-1/2)^4$, notjustitssimplifiedvalue$5/256$), 1forstating/usingthattheseriesisalternating(sothealte
$1/50$.

Solution 6

(c) Mark scheme

- Differentiating the Maclaurin series for f term by term, the general term of the Maclaurin series for f' is $\frac{(n+1)nx^{n-1}}{n^2 6^n} = \frac{(n+1)x^{n-1}}{n \cdot 6^n}$.*Becauseterm* − *by* −
termdifferentiationofapowerseriesdoesnotchangeitsradiusofconvergence, *andtheradiusofconverg*
1*forthecorrectgeneralterm*$(n+1)x^{n-1}/(n \cdot 6^n)$*oranequivalentunsimplifiedform*(*e.g.*$(n+$
$1)nx^{n-1}/(n^2 6^n)$), 1*forthecorrectradiusofconvergence*6, *supported*(*thispointcanbeearnedindepen*
testorroot − *testcomputation*).

Solution 6

(d) Mark scheme

- For g(x)=$\sum_{n=1}^{\infty} \frac{(n+1)x^{2n}}{n^2 3^n}, applytheratiotest: \left|\frac{(n+2)x^{2n+2}/((n+1)^2 3^{n+1})}{(n+1)x^{2n}/(n^2 3^n)}\right| = \left|\frac{(n+2)x^{2n+2}}{(n+1)^2 3^{n+1}} \cdot \frac{n^2 3^n}{(n+1)x^{2n}}\right|. Takingthelimit: \lim_{n\to\infty}\left|\frac{(n+2)n^2}{(n+1)^3}\right| \cdot \left|\frac{x^2}{3}\right| = \left|\frac{x^2}{3}\right|. Setting \left|\frac{x^2}{3}\right| < 1 givesx^2 < 3, i.e. |x| < \sqrt{3}. SotheradiusofconvergenceoftheMaclaurinseriesforgis \sqrt{3}. Points: 1 forcorrectlysettinguptheratioofconsecutiveterms(withorwithoutabsolutevalues; onceearnedthis$

Question 6

2022 ▪ Q6

The function f is defined by the power series $f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + \frac{(-1)^n x^{2n+1}}{2n+1} + \cdots$ for all real numbers x for which the series converges.

(a) Using the ratio test, find the interval of convergence of the power series for f. Justify your answer.

(b) Show that $\left| f\left(\frac{1}{2}\right) - \frac{1}{2} \right| < \frac{1}{10}$. Justify your answer.

(c) Write the first four nonzero terms and the general term for an infinite series that represents $f'(x)$.

(d) Use the result from part (c) to find the value of $f'\left(\frac{1}{6}\right)$.

Solution 6

(a) Mark scheme

- Ratio test: $\lim_{n\to\infty}\left|\dfrac{(-1)^{n+1}x^{2n+3}/(2n+3)}{(-1)^n x^{2n+1}/(2n+1)}\right| = \lim_{n\to\infty}\left|x^2\dfrac{2n+1}{2n+3}\right| = |x^2| = x^2$. Need $x^2 < 1 \Rightarrow -1 < x < 1$ (interior of interval). At $x = -1$: series is $-1 + \frac{1}{3} - \frac{1}{5} + \cdots$, alternating with terms decreasing in absolute value to 0, converges by the Alternating Series Test. At $x = 1$: series is $1 - \frac{1}{3} + \frac{1}{5} - \cdots$, also alternating with terms decreasing to 0, converges by AST. So the interval of convergence is $-1 \leq x \leq 1$. 4 points: 1 for setting up the ratio, 1 for identifying the interior $-1 < x < 1$ from the limit, 1 for considering both endpoints ($x = \pm 1$), 1 for correct analysis at both endpoints and stating the full interval $-1 \leq x \leq 1$.

Solution 6

(b) Mark scheme

- $f(1/2)$ is an alternating series with first term $\frac{1}{2}$ and terms decreasing in absolute value to 0. By the alternating series error bound, $\left|f(\frac{1}{2}) - \frac{1}{2}\right| < |\text{second term}| = \left|\dfrac{(-1)^1(1/2)^3}{3}\right| = \dfrac{1}{24} < \dfrac{1}{10}$. 2 points: 1 for correctly using $x = \frac{1}{2}$ in the second term of the series, 1 for stating the series is alternating with terms decreasing to 0 and presenting the inequality Error $< \frac{1}{24} < \frac{1}{10}$ (stating Error $= 1/24$ alone does not earn this point).

Solution 6

(c)

Mark scheme

- $f'(x) = 1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n} + \cdots$ (term-by-term derivative of the series for f). 2 points: 1 for presenting the first four nonzero terms $(1, -x^2, x^4, -x^6)$, 1 for the correct general term $(-1)^n x^{2n}$.

Solution 6

(d) Mark scheme

- $f(1/6) = 1 - (1/6)^2 + (1/6)^4 - (1/6)^6 + \cdots$ is a geometric series with first term $a = 1$ and common ratio $r = -1/36$.
  $f(1/6) = \dfrac{a}{1-r} = \dfrac{1}{1-(-1/36)} = \dfrac{1}{37/36} = \dfrac{36}{37}$. 1 point for the correct answer $36/37$, contingent on correctly identifying the series from part (c) as geometric.

Question 6

2021 ▪ Q6

The function g has derivatives of all orders for all real numbers. The Maclaurin series for g is given by $g(x) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2e^n + 3}$ on its interval of convergence.

(a) State the conditions necessary to use the integral test to determine convergence of the series $\displaystyle\sum_{n=0}^{\infty} \frac{1}{e^n}$.

Use the integral test to show that $\displaystyle\sum_{n=0}^{\infty} \frac{1}{e^n}$ converges.

Question 6

2021 ▪ Q6

The function g has derivatives of all orders for all real numbers. The Maclaurin series for g is given by $g(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2e^n + 3}$ on its interval of convergence.

(b) Use the limit comparison test with the series $\sum_{n=0}^{\infty} \frac{1}{e^n}$ to show that the series

$$g(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2e^n + 3}$$ converges absolutely.

Question 6

2021 ▪ Q6

The function g has derivatives of all orders for all real numbers. The Maclaurin series for g is given by $g(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2e^n + 3}$ on its interval of convergence.

(c) Determine the radius of convergence of the Maclaurin series for g.

Question 6

2021 ▪ Q6

The function g has derivatives of all orders for all real numbers. The Maclaurin series for g is given by $g(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2e^n + 3}$ on its interval of convergence.

(d) The first two terms of the series $g(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2e^n + 3}$ are used to approximate $g(1)$.

Use the alternating series error bound to determine an upper bound on the error of the approximation.

Solution 6

(a) Mark scheme

- Conditions for the integral test: e^{-x} must be positive, decreasing, and continuous on $[0, \infty)$ — all hold.
  $\int_0^\infty e^{-x}\,dx = \lim_{b\to\infty} \int_0^b e^{-x}\,dx = \lim_{b\to\infty} (-e^{-b} + e^0) = 0 + 1 = 1$, which is finite, so the improper integral converges. Therefore, by the integral test, $\sum_{n=0}^{\infty} \frac{1}{e^n}$ converges. (1 pt: states all three conditions — positive, decreasing, continuous; 1 pt: correctly writes the improper integral or an equivalent limit; 1 pt: correctly evaluates the limit using proper limit notation, finds e^0 or 1, and concludes convergence — a response that only uses a geometric-series approach earns 0 points here.)

Solution 6

(b) Mark scheme

- $\lim_{n\to\infty} \frac{1/e^n}{|(-1)^n/(2e^n+3)|} = \lim_{n\to\infty} \frac{2e^n+3}{e^n} = 2$. This limit exists and is positive, so by the limit comparison test (using absolute values), since $\sum_{n=0}^{\infty} \frac{1}{e^n}$ converges, $\sum_{n=0}^{\infty} \left|\frac{(-1)^n}{2e^n+3}\right|$ also converges. Therefore $g(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2e^n+3}$ converges absolutely. (1 pt: correctly sets up the limit comparison with limit notation, with or without absolute values explicit at setup; 1 pt: correctly evaluates the limit as a finite positive number and concludes absolute convergence specifically of $g(1)$ — requires absolute-value reasoning to have appeared.)

Solution 6

(c) Mark scheme

- Ratio test: $\left|\dfrac{a_{n+1}x^{n+1}}{a_n x^n}\right| = \dfrac{2e^n+3}{2e^{n+1}+3}\,|x| \to \dfrac{1}{e}|x|$ as $n \to \infty$. Setting $\dfrac{1}{e}|x| < 1$ gives $|x| < e$. The radius of convergence is $R = e$. (1 pt: correctly sets up the ratio of consecutive terms; 1 pt: correctly computes the limit of the ratio as $|x|/e$ [finite, nonzero coefficient]; 1 pt: explicit final answer $R = e$ from solving the inequality — presenting only an interval like $-e < x < e$ without identifying R does not earn this point.)

Solution 6

(d) Mark scheme

- The terms of the alternating series $g(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2e^n+3}$ decrease in magnitude to 0, so the alternating series error bound applies: the error from using the first two terms ($n = 0, 1$) to approximate $g(1)$ is bounded by the absolute value of the next (third, $n = 2$) term: Error $\leq \left| \dfrac{(-1)^2}{2e^2+3} \right| = \dfrac{1}{2e^2+3}$. (1 pt: correct upper bound $\frac{1}{2e^2+3}$.)

Question 6

2018 ▪ Q6

The Maclaurin series for $\ln(1+x)$ is given by

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n+1}\frac{x^n}{n} + \cdots .$$

On its interval of convergence, this series converges to $\ln(1+x)$. Let f be the function defined by

$$f(x) = x \ln\left(1 + \frac{x}{3}\right).$$

(a) Write the first four nonzero terms and the general term of the Maclaurin series for f.

Question 6

(b) Determine the interval of convergence of the Maclaurin series for f. Show the work that leads to your answer.

(c) Let $P_4(x)$ be the fourth-degree Taylor polynomial for f about $x = 0$. Use the alternating series error bound to find an upper bound for $|P_4(2) - f(2)|$.

Solution 6

(a) Mark scheme

- The first four nonzero terms of the Maclaurin series for f are $\frac{x^2}{3} - \frac{x^3}{2 \cdot 3^2} + \frac{x^4}{3 \cdot 3^3} - \frac{x^5}{4 \cdot 3^4}$. The general term is $(-1)^{n+1} \frac{x^{n+1}}{n \cdot 3^n}$. Points: 1 for the correct first four nonzero terms, 1 for the correct general term.

Solution 6

(b) Mark scheme

- Using the ratio test: $\lim_{n\to\infty}\left|\frac{(-1)^{n+2}x^{n+2}/((n+1)3^{n+1})}{(-1)^{n+1}x^{n+1}/(n\cdot 3^n)}\right| = \lim_{n\to\infty}\left|\frac{-x}{3}\cdot\frac{n}{n+1}\right| = \left|\frac{x}{3}\right|$. This is < 1 for $|x| < 3$, so the radius of convergence is 3. (Equivalently, using the known radius of convergence 1 for the Maclaurin series of $\ln(1+x)$ and $f(x) = x\ln(1+x/3)$, substitute $x/3$ to also get radius 3.) Checking the endpoints: at $x = -3$, the series is $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(-3)^{n+1}}{n\cdot 3^n} = \sum_{n=1}^{\infty}\frac{3}{n}$, which diverges by comparison to the harmonic series. At $x = 3$, the series is $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{3^{n+1}}{n\cdot 3^n} = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{3}{n}$, which converges by the alternating series test. The interval of convergence of the

Solution 6

Maclaurin series for f is $-3 < x \leq 3$. Points: 1 for setting up the ratio test, 1 for correctly computing the limit of the ratio, 1 for the correct radius of convergence (3), 1 for considering both endpoints $x = \pm 3$, 1 for the correct analysis and interval of convergence ($-3 < x \leq 3$).

Solution 6

(c)

Mark scheme

- By the alternating series error bound, an upper bound for $|P_4(2) - f(2)|$ is the magnitude of the next (fifth-degree) term of the alternating series: $|P_4(2) - f(2)| < \left|-\dfrac{2^5}{4 \cdot 3^4}\right| = \dfrac{8}{81}$. Points: 1 for using the fifth-degree term as the error bound, 1 for the correct answer $(8/81)$.

Question 6

2016 ▪ Q6

The function f has a Taylor series about $x = 1$ that converges to $f(x)$ for all x in the interval of convergence. It is known that $f(1) = 1$, $f'(1) = -\frac{1}{2}$, and the nth derivative of f at $x = 1$ is given by $f^{(n)}(1) = (-1)^n \frac{(n-1)!}{2^n}$ for $n \geq 2$.

(a) Write the first four nonzero terms and the general term of the Taylor series for f about $x = 1$.

(b) The Taylor series for f about $x = 1$ has a radius of convergence of 2. Find the interval of convergence. Show the work that leads to your answer.

(c) The Taylor series for f about $x = 1$ can be used to represent $f(1.2)$ as an alternating series. Use the first three nonzero terms of the alternating series to approximate $f(1.2)$.

Question 6

(d) Show that the approximation found in part (c) is within 0.001 of the exact value of $f(1.2)$.

Solution 6

(a) Mark scheme

- Using $f(1)=1$, $f'(1)=-\frac{1}{2}$, and $f^{(n)}(1)=(-1)^n\frac{(n-1)!}{2^n}$ for $n\geq 2$ (so $f''(1)=\frac{1}{2^2}=\frac{1}{4}$, $f'''(1)=-\frac{2}{2^3}=-\frac{1}{4}$): $f(x)=1-\frac{1}{2}(x-1)+\frac{1}{8}(x-1)^2-\frac{1}{24}(x-1)^3+\cdots+\frac{(-1)^n}{n\cdot 2^n}(x-1)^n+\cdots$. Award 1 point for the first two terms (1 and $-\frac{1}{2}(x-1)$), 1 point for the third term $\frac{1}{8}(x-1)^2$, 1 point for the fourth term $-\frac{1}{24}(x-1)^3$, and 1 point for the correct general term $\frac{(-1)^n}{n\cdot 2^n}(x-1)^n$.

Solution 6

(b)

Mark scheme

- Radius of convergence $R = 2$ about center $x = 1$, so the series converges on the open interval $(-1, 3)$. At $x = -1$: the series becomes $1 + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$, which (up to overall sign) is the harmonic series and diverges. At $x = 3$: the series becomes $1 - 1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \cdots$, the alternating harmonic series, which converges. Therefore the interval of convergence is $-1 < x \leq 3$. Award 1 point for testing/identifying both endpoints, and 1 point for the correct analysis and final interval.

Solution 6

(c)

Mark scheme

- $f(1.2) \approx 1 - \frac{1}{2}(0.2) + \frac{1}{8}(0.2)^2 = 1 - 0.1 + 0.005 = 0.905$. Award 1 point for the correct approximation 0.905.

Solution 6

(d)

Mark scheme

- The series for $f(1.2)$ alternates with terms decreasing in magnitude to 0, so the Alternating Series Error Bound applies: $|f(1.2) - T_2(1.2)| \leq |\text{next term}| = \left|-\frac{1}{2^3 \cdot 3}(0.2)^3\right| = \frac{1}{3000} \approx 0.000333 \leq 0.001$. Award 1 point for citing the correct error-bound/next-term form, and 1 point for the numeric analysis showing it is ≤ 0.001.

Question 6

2015 ▪ Q6

The Maclaurin series for a function f is given by
$\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{n} x^n = x - \frac{3}{2}x^2 + 3x^3 - \cdots + \frac{(-3)^{n-1}}{n} x^n + \cdots$ and converges to $f(x)$ for $|x| < R$, where R is the radius of convergence of the Maclaurin series.

(a) Use the ratio test to find R.

(b) Write the first four nonzero terms of the Maclaurin series for f', the derivative of f. Express f' as a rational function for $|x| < R$.

(c) Write the first four nonzero terms of the Maclaurin series for e^x. Use the Maclaurin series for e^x to write the third-degree Taylor polynomial for $g(x) = e^x f(x)$ about $x = 0$.

Solution 6

(a) Mark scheme

- Let a_n be the nth term of the Maclaurin series. $\frac{a_{n+1}}{a_n} = \frac{(-3)^n x^{n+1}}{n+1} \cdot \frac{n}{(-3)^{n-1} x^n} = \frac{-3n}{n+1} x$. $\lim_{n\to\infty} \left| \frac{-3n}{n+1} x \right| = 3|x|$. $3|x| < 1 \Rightarrow |x| < \frac{1}{3}$. So the radius of convergence is $R = \frac{1}{3}$. Award 1 point for setting up the ratio $|a_{n+1}/a_n|$, 1 point for correctly computing the limit of the ratio ($3|x|$), and 1 point for determining $R = 1/3$.

Solution 6

(b)

Mark scheme

- The first four nonzero terms of the Maclaurin series for f' are $1 - 3x + 9x^2 - 27x^3$ (term-by-term differentiation of the given series for f). As a rational function, $f'(x) = \frac{1}{1-(-3x)} = \frac{1}{1+3x}$ for $|x| < R$. Award 2 points for the correct first four nonzero terms, and 1 point for the correct rational function expression.

Solution 6

(c) Mark scheme

- The first four nonzero terms of the Maclaurin series for e^x are $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$. Multiplying by the Maclaurin series for f: $\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\left(x - \frac{3}{2}x^2 + 3x^3 - \cdots\right) = x - \frac{1}{2}x^2 + 2x^3 + \cdots$. So the third-degree Taylor polynomial for $g(x) = e^x f(x)$ about $x = 0$ is $T_3(x) = x - \frac{1}{2}x^2 + 2x^3$. Award 1 point for the correct first four nonzero terms of the Maclaurin series for e^x, and 2 points for the correct third-degree Taylor polynomial for g.

Question 6

2014 ▪ Q6

The Taylor series for a function f about $x = 1$ is given by $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2}{n}(x-1)^n$ and converges to $f(x)$ for $|x-1| < R$, where R is the radius of convergence of the Taylor series.

(a) Find the value of R.

(b) Find the first three nonzero terms and the general term of the Taylor series for f', the derivative of f, about $x = 1$.

(c) The Taylor series for f' about $x = 1$, found in part (b), is a geometric series. Find the function f' to which the series converges for $|x-1| < R$. Use this function to determine f for $|x-1| < R$.