Past-paper walk-through

Lagrange Error Bound ▪ 10.12

eXercise

Questions in this set

- 2026 Q6
- 2025 Q5
- 2023 Q6

Reading the mark scheme

- **B1** a mark that stands on its own – the point is either made or not.
- **M1** a *method* mark: the right approach, even if the number is wrong.
- **A1** an *answer* mark, and it usually needs its M mark first.
- **C1** a working mark on the way to the answer.
- // separates answers the examiner accepts equally.
- **ecf** = error carried forward: a later part is marked on your own earlier answer.
- **owtte** = "or words to that effect" – the wording need not match.

Question 6

2026 ▪ Q6

The Maclaurin series for a function g is given by

$$\sum_{n=0}^{\infty} 2\left(\frac{-x}{5}\right)^n = 2 - \frac{2}{5}x + \frac{2}{25}x^2 - \frac{2}{125}x^3 + \frac{2}{625}x^4 - \cdots + 2\left(\frac{-x}{5}\right)^n + \cdots$$

and converges to $g(x)$ for $|x| < 5$.

(a) For $x > 0$, the Maclaurin series for g is an alternating geometric series. Find $g(3)$. Show the work that leads to your answer.

Question 6

2026 ▪ Q6

The Maclaurin series for a function g is given by

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and converges to $g(x)$ for $|x| < 5$.

(b) The function f is defined by $f(x) = g'(x)$. Write the first four nonzero terms of the Maclaurin series for f.

Question 6

2026 ▪ Q6

The Maclaurin series for a function g is given by

$$\sum_{n=0}^{\infty} 2\left(\frac{-x}{5}\right)^n = 2 - \frac{2}{5}x + \frac{2}{25}x^2 - \frac{2}{125}x^3 + \frac{2}{625}x^4 - \cdots + 2\left(\frac{-x}{5}\right)^n + \cdots$$

and converges to $g(x)$ for $|x| < 5$.

(c) The second-degree Taylor polynomial for f about $x = 0$ is used to approximate $f\left(\frac{5}{2}\right)$ as $-\frac{3}{10}$, where f is the function defined in part B. Justify that $\left|f\left(\frac{5}{2}\right) - \left(-\frac{3}{10}\right)\right| \leq \frac{1}{5}$.

Question 6

2026 ▪ Q6

The Maclaurin series for a function g is given by

$$\sum_{n=0}^{\infty} 2\left(\frac{-x}{5}\right)^n = 2 - \frac{2}{5}x + \frac{2}{25}x^2 - \frac{2}{125}x^3 + \frac{2}{625}x^4 - \cdots + 2\left(\frac{-x}{5}\right)^n + \cdots$$

and converges to $g(x)$ for $|x| < 5$.

(d)(i) The function h is defined by $h(x) = 25g(x) - 2e^x$. Write the first three nonzero terms of the Maclaurin series for e^x.

(d)(ii) Write the first three nonzero terms of the Maclaurin series for h.

Question 5

2025 ▪ Q5

Let $y = f(x)$ be the particular solution to the differential equation $\dfrac{dy}{dx} = (3 - x)y^2$ with initial condition $f(1) = -1$.

(a) Find $f''(1)$, the value of $\dfrac{d^2y}{dx^2}$ at the point $(1, -1)$. Show the work that leads to your answer.

(b) Write the second-degree Taylor polynomial for f about $x = 1$.

(c) The second-degree Taylor polynomial for f about $x = 1$ is used to approximate $f(1.1)$. Given that $|f'''(x)| \leq 60$ for all x in the interval $1 \leq x \leq 1.1$, use the Lagrange error bound to show that this approximation differs from $f(1.1)$ by at most 0.01.

(d) Use Euler's method, starting at $x = 1$ with two steps of equal size, to approximate $f(1.4)$. Show the work that leads to your answer.

Solution 5

(a) Mark scheme

- $\frac{dy}{dx} = (3-x)y^2$. Differentiating implicitly (product rule then chain rule): $\frac{d^2y}{dx^2} = -y^2 + (3-x)\cdot 2y\frac{dy}{dx}$. At $(1,-1)$: $\left.\frac{dy}{dx}\right|_{(1,-1)} = (3-1)(-1)^2 = 2$. Then $f''(1) = \left.\frac{d^2y}{dx^2}\right|_{(1,-1)} = -(-1)^2 + (3-1)(2)(-1)(2) = -1-8 = -9$. (1 pt: correct product rule; 1 pt: correct chain rule; 1 pt: correct value $f''(1) = -9$.)

Solution 5

(b)

Mark scheme

- Using $f(1) = -1$, $f'(1) = 2$, $f''(1) = -9$: $P_2(x) = -1 + 2(x-1) - \frac{9}{2}(x-1)^2$.
  (1 pt: correct first two terms, constant + linear; 1 pt: correct remaining quadratic term, presented as a power of $(x-1)$.)

Solution 5

(c) Mark scheme

- Lagrange error bound:
  $|f(1.1) - P_2(1.1)| \leq \dfrac{\max_{1 \leq x \leq 1.1} |f'''(x)|}{3!} |1.1 - 1|^3 \leq \dfrac{60}{6}(0.1)^3 = 0.01$. This shows the approximation differs from $f(1.1)$ by at most 0.01. (1 pt: correct form of the error bound presented, e.g. $\frac{\max |f'''(x)|}{3!} |1.1 - 1|^3$ or $\frac{60}{6}(0.1)^3$; 1 pt: explicitly connects the computed bound to 0.01, e.g. 'Error ≤ 0.01'.)

Solution 5

(d) Mark scheme

- Euler's method with two steps of size 0.2 from $x = 1$: Step 1: $f(1.2) \approx f(1) + 0.2 \cdot \frac{dy}{dx}\Big|_{(1,-1)} = -1 + 0.2(2) = -0.6$. Step 2: $f(1.4) \approx f(1.2) + 0.2 \cdot \frac{dy}{dx}\Big|_{(1.2,-0.6)} = -0.6 + 0.2(3 - 1.2)(-0.6)^2 = -0.6 + 0.1296 = -0.4704$. So $f(1.4) \approx -0.47$. (1 pt: correct first step of Euler's method with correct initial condition, step size, and derivative expression; 1 pt: correct answer with supporting work, ≈ -0.47.)

Question 6

2023 ▪ Q6

The function f has derivatives of all orders for all real numbers. It is known that $f(0) = 2$, $f'(0) = 3$, $f''(x) = -f(x^2)$, and $f'''(x) = -2x \cdot f'(x^2)$.

(a) Find $f^{(4)}(x)$, the fourth derivative of f with respect to x. Write the fourth-degree Taylor polynomial for f about $x = 0$. Show the work that leads to your answer.

(b) The fourth-degree Taylor polynomial for f about $x = 0$ is used to approximate $f(0.1)$. Given that $\left|f^{(5)}(x)\right| \leq 15$ for $0 \leq x \leq 0.5$, use the Lagrange error bound to show that this approximation is within $\dfrac{1}{10^5}$ of the exact value of $f(0.1)$.

(c) Let g be the function such that $g(0) = 4$ and $g'(x) = e^x f(x)$. Write the second-degree Taylor polynomial for g about $x = 0$.

Solution 6

(a) Mark scheme

- Given $f''(x) = -f(x^2)$ and $f'''(x) = -2x \cdot f'(x^2)$, differentiate again using the product rule: $f^{(4)}(x) = -2 \cdot f'(x^2) + (-2x) \cdot f''(x^2) \cdot 2x$. Using $f(0) = 2$, $f'(0) = 3$: $f''(0) = -f(0) = -2$; $f'''(0) = -2(0) \cdot f'(0) = 0$; $f^{(4)}(0) = -2 \cdot f'(0) + 0 \cdot f''(0) \cdot 0 = -2 \cdot 3 + 0 = -6$. The fourth-degree Taylor polynomial for f about $x = 0$ is $T_4(x) = 2 + 3x + \frac{-2}{2!}x^2 + \frac{0}{3!}x^3 + \frac{-6}{4!}x^4 = 2 + 3x - x^2 - \frac{1}{4}x^4$. (1 point for the correct form of the product rule, $f^{(4)}(x) = -2f'(x^2) + (-2x)f''(x^2) \cdot 2x$; 1 point for the fully correct expression of $f^{(4)}(x)$; 1 point for the first two terms of the polynomial, $2 + 3x$; 1 point for the remaining correct terms $-x^2 - \frac{1}{4}x^4$ — a

Solution 6

polynomial with a nonzero third-degree term, any term of degree greater than 4, or a trailing '$+ \ldots$' forfeits this point.)

Solution 6

(b) Mark scheme

- By the Lagrange error bound: $|T_4(0.1) - f(0.1)| \leq \frac{\max_{0\leq x\leq 0.1}|f^{(5)}(x)|}{5!} \cdot (0.1)^5 \leq \frac{15}{120} \cdot \frac{1}{10^5} \leq \frac{1}{10^5}$. (1 point for the correct form of the error bound $\frac{\max|f^{(5)}(x)|}{5!} \cdot (0.1)^5$ or $\frac{15}{5!}(0.1)^5$ — subsequent simplification errors do not forfeit this point but do forfeit the second; 1 point for communicating the full inequality Error $\leq \frac{15}{5!}(0.1)^5 \leq \frac{1}{10^5}$ — stating only Error $= \frac{15}{5!}(0.1)^5$ or Error $= \frac{1}{10^5}$ does not earn this point.)

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(c)

Mark scheme

- Given $g(0) = 4$, $g'(x) = e^x f(x)$: $g''(x) = e^x \cdot f(x) + e^x \cdot f'(x)$. At $x = 0$: $g'(0) = e^0 f(0) = 2$, and $g''(0) = e^0 f(0) + e^0 f'(0) = 2 + 3 = 5$. The second-degree Taylor polynomial for g about $x = 0$ is $T_2(x) = 4 + 2x + \frac{5}{2}x^2$. (1 point for $g''(x) = e^x f(x) + e^x f'(x)$, or $g''(0) = f(0) + f'(0)$, or the equivalent series-product approach yielding $e^x f(x) = 2 + 5x + \cdots$; 1 point for the first two terms $4 + 2x$ of the polynomial; 1 point for the fully correct polynomial $4 + 2x + \frac{5}{2}x^2$ with supporting work — a presented polynomial including any term of degree greater than 2, or a trailing '+...', does not earn this point.)