Past-paper walk-through

Finding Taylor Polynomial Approximations of Functions ▪ 10.11

eXercise

Questions in this set

- 2025 Q5
- 2023 Q6
- 2021 Q5
- 2019 Q6
- 2016 Q6
- 2015 Q6

Reading the mark scheme

- **B1** a mark that stands on its own – the point is either made or not.
- **M1** a *method* mark: the right approach, even if the number is wrong.
- **A1** an *answer* mark, and it usually needs its M mark first.
- **C1** a working mark on the way to the answer.
- // separates answers the examiner accepts equally.
- **ecf** = error carried forward: a later part is marked on your own earlier answer.
- **owtte** = "or words to that effect" – the wording need not match.

Question 5

2025 ▪ Q5

Let $y = f(x)$ be the particular solution to the differential equation $\frac{dy}{dx} = (3 - x)y^2$ with initial condition $f(1) = -1$.

(a) Find $f''(1)$, the value of $\frac{d^2y}{dx^2}$ at the point $(1, -1)$. Show the work that leads to your answer.

(b) Write the second-degree Taylor polynomial for f about $x = 1$.

(c) The second-degree Taylor polynomial for f about $x = 1$ is used to approximate $f(1.1)$. Given that $|f'''(x)| \leq 60$ for all x in the interval $1 \leq x \leq 1.1$, use the Lagrange error bound to show that this approximation differs from $f(1.1)$ by at most 0.01.

(d) Use Euler's method, starting at $x = 1$ with two steps of equal size, to approximate $f(1.4)$. Show the work that leads to your answer.

Solution 5

(a) Mark scheme

- $\dfrac{dy}{dx} = (3-x)y^2$. Differentiating implicitly (product rule then chain rule): $\dfrac{d^2y}{dx^2} = -y^2 + (3-x)\cdot 2y\dfrac{dy}{dx}$. At $(1,-1)$: $\left.\dfrac{dy}{dx}\right|_{(1,-1)} = (3-1)(-1)^2 = 2$. Then $f''(1) = \left.\dfrac{d^2y}{dx^2}\right|_{(1,-1)} = -(-1)^2 + (3-1)(2)(-1)(2) = -1-8 = -9$. (1 pt: correct product rule; 1 pt: correct chain rule; 1 pt: correct value $f''(1) = -9$.)

Solution 5

(b)

Mark scheme

- Using $f(1) = -1$, $f'(1) = 2$, $f''(1) = -9$: $P_2(x) = -1 + 2(x-1) - \frac{9}{2}(x-1)^2$. (1 pt: correct first two terms, constant + linear; 1 pt: correct remaining quadratic term, presented as a power of $(x-1)$.)

Solution 5

(c) Mark scheme

- Lagrange error bound: $|f(1.1) - P_2(1.1)| \leq \frac{\max_{1 \leq x \leq 1.1} |f'''(x)|}{3!} |1.1 - 1|^3 \leq \frac{60}{6}(0.1)^3 = 0.01$. This shows the approximation differs from $f(1.1)$ by at most 0.01. (1 pt: correct form of the error bound presented, e.g. $\frac{\max |f'''(x)|}{3!}|1.1-1|^3$ or $\frac{60}{6}(0.1)^3$; 1 pt: explicitly connects the computed bound to 0.01, e.g. 'Error ≤ 0.01'.)

Solution 5

(d) Mark scheme

- Euler's method with two steps of size 0.2 from $x = 1$: Step 1: $f(1.2) \approx f(1) + 0.2 \cdot \left.\frac{dy}{dx}\right|_{(1,-1)} = -1 + 0.2(2) = -0.6$. Step 2: $f(1.4) \approx f(1.2) + 0.2 \cdot \left.\frac{dy}{dx}\right|_{(1.2,-0.6)} = -0.6 + 0.2(3 - 1.2)(-0.6)^2 = -0.6 + 0.1296 = -0.4704$. So $f(1.4) \approx -0.47$. (1 pt: correct first step of Euler's method with correct initial condition, step size, and derivative expression; 1 pt: correct answer with supporting work, ≈ -0.47.)

Question 6

2023 ▪ Q6

The function f has derivatives of all orders for all real numbers. It is known that $f(0) = 2$, $f'(0) = 3$, $f''(x) = -f(x^2)$, and $f'''(x) = -2x \cdot f'(x^2)$.

(a) Find $f^{(4)}(x)$, the fourth derivative of f with respect to x. Write the fourth-degree Taylor polynomial for f about $x = 0$. Show the work that leads to your answer.

(b) The fourth-degree Taylor polynomial for f about $x = 0$ is used to approximate $f(0.1)$. Given that $\left|f^{(5)}(x)\right| \leq 15$ for $0 \leq x \leq 0.5$, use the Lagrange error bound to show that this approximation is within $\dfrac{1}{10^5}$ of the exact value of $f(0.1)$.

(c) Let g be the function such that $g(0) = 4$ and $g'(x) = e^x f(x)$. Write the second-degree Taylor polynomial for g about $x = 0$.

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(a) Mark scheme

- Given $f''(x) = -f(x^2)$ and $f'''(x) = -2x \cdot f'(x^2)$, differentiate again using the product rule: $f^{(4)}(x) = -2 \cdot f'(x^2) + (-2x) \cdot f''(x^2) \cdot 2x$. Using $f(0) = 2$, $f'(0) = 3$: $f''(0) = -f(0) = -2$; $f'''(0) = -2(0) \cdot f'(0) = 0$; $f^{(4)}(0) = -2 \cdot f'(0) + 0 \cdot f''(0) \cdot 0 = -2 \cdot 3 + 0 = -6$. The fourth-degree Taylor polynomial for f about $x = 0$ is $T_4(x) = 2 + 3x + \frac{-2}{2!}x^2 + \frac{0}{3!}x^3 + \frac{-6}{4!}x^4 = 2 + 3x - x^2 - \frac{1}{4}x^4$. (1 point for the correct form of the product rule, $f^{(4)}(x) = -2f'(x^2) + (-2x)f''(x^2) \cdot 2x$; 1 point for the fully correct expression of $f^{(4)}(x)$; 1 point for the first two terms of the polynomial, $2 + 3x$; 1 point for the remaining correct terms $-x^2 - \frac{1}{4}x^4$ — a

Solution 6

polynomial with a nonzero third-degree term, any term of degree greater than 4, or a trailing '$+ \ldots$' forfeits this point.)

Solution 6

(b) Mark scheme

- By the Lagrange error bound: $|T_4(0.1) - f(0.1)| \leq \dfrac{\max_{0\leq x\leq 0.1}|f^{(5)}(x)|}{5!} \cdot (0.1)^5 \leq \dfrac{15}{120} \cdot \dfrac{1}{10^5} \leq \dfrac{1}{10^5}$. (1 point for the correct form of the error bound $\frac{\max|f^{(5)}(x)|}{5!} \cdot (0.1)^5$ or $\frac{15}{5!}(0.1)^5$ — subsequent simplification errors do not forfeit this point but do forfeit the second; 1 point for communicating the full inequality Error $\leq \frac{15}{5!}(0.1)^5 \leq \frac{1}{10^5}$ — stating only Error $= \frac{15}{5!}(0.1)^5$ or Error $= \frac{1}{10^5}$ does not earn this point.)

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(c)

Mark scheme

- Given $g(0) = 4$, $g'(x) = e^x f(x)$: $g''(x) = e^x \cdot f(x) + e^x \cdot f'(x)$. At $x = 0$: $g'(0) = e^0 f(0) = 2$, and $g''(0) = e^0 f(0) + e^0 f'(0) = 2 + 3 = 5$. The second-degree Taylor polynomial for g about $x = 0$ is $T_2(x) = 4 + 2x + \frac{5}{2}x^2$. (1 point for $g''(x) = e^x f(x) + e^x f'(x)$, or $g''(0) = f(0) + f'(0)$, or the equivalent series-product approach yielding $e^x f(x) = 2 + 5x + \cdots$; 1 point for the first two terms $4 + 2x$ of the polynomial; 1 point for the fully correct polynomial $4 + 2x + \frac{5}{2}x^2$ with supporting work — a presented polynomial including any term of degree greater than 2, or a trailing '+...', does not earn this point.)

Question 5

2021 ▪ Q5

Let $y = f(x)$ be the particular solution to the differential equation $\dfrac{dy}{dx} = y \cdot (x \ln x)$ with initial condition $f(1) = 4$. It can be shown that $f''(1) = 4$.

(a) Write the second-degree Taylor polynomial for f about $x = 1$. Use the Taylor polynomial to approximate $f(2)$.

(b) Use Euler's method, starting at $x = 1$ with two steps of equal size, to approximate $f(2)$. Show the work that leads to your answer.

(c) Find the particular solution $y = f(x)$ to the differential equation $\dfrac{dy}{dx} = y \cdot (x \ln x)$ with initial condition $f(1) = 4$.

Solution 5

(a)

Mark scheme

- $f'(1) = \left.\dfrac{dy}{dx}\right|_{(1,4)} = 4 \cdot (1 \ln 1) = 0$ (and it is given that $f''(1) = 4$).
  Second-degree Taylor polynomial about $x = 1$:
  $P(x) = f(1) + f'(1)(x-1) + \dfrac{f''(1)}{2!}(x-1)^2 = 4 + 0(x-1) + 2(x-1)^2$. Using it: $f(2) \approx 4 + 2(2-1)^2 = 6$. (1 pt: correct Taylor polynomial $4 + 2(x-1)^2$ [or equivalent, must be centered at $x = 1$]; 1 pt: approximation $f(2) \approx 6$.)

Solution 5

(b)

Mark scheme

- Euler's method with $\Delta x = 0.5$ starting at $(1, 4)$: Step 1: $f(1.5) \approx f(1) + 0.5 \cdot \left.\frac{dy}{dx}\right|_{(1,4)} = 4 + 0.5 \cdot [4 \cdot (1 \ln 1)] = 4 + 0 = 4$. Step 2: $f(2) \approx f(1.5) + 0.5 \cdot \left.\frac{dy}{dx}\right|_{(1.5,4)} = 4 + 0.5 \cdot [4 \cdot (1.5 \ln 1.5)] = 4 + 3 \ln 1.5$ (≈ 5.216). (1 pt: two correct Euler steps of size 0.5 using $dy/dx = y(x \ln x)$ and $f(1) = 4$, at most one error; 1 pt: final answer $4 + 3 \ln 1.5$ — requires the first point.)

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(c) Mark scheme

- Separate variables: $\frac{1}{y}\,dy = x \ln x \, dx$. Integrate the right side by parts ($u = \ln x$, $dv = x\,dx$): $\int x \ln x \, dx = \frac{x^2}{2}\ln x - \frac{x^2}{4} + C$. Left side: $\int \frac{1}{y}\,dy = \ln|y|$. So $\ln|y| = \frac{x^2}{2}\ln x - \frac{x^2}{4} + C$. Apply $f(1) = 4$: $\ln 4 = 0 - \frac{1}{4} + C \Rightarrow C = \ln 4 + \frac{1}{4}$. Solve for y: $y = e^{\frac{x^2}{2}\ln x - \frac{x^2}{4} + \ln 4 + \frac{1}{4}}$, valid for $x > 0$. (1 pt: correctly separates variables; 1 pt: correct antiderivative of $x \ln x$ via integration by parts; 1 pt: correct antiderivative $\ln|y|$ for $\frac{1}{y}$; 1 pt: correctly finds the constant of integration

Solution 5

using $f(1) = 4$ — requires the separation point and at least one antiderivative point; 1 pt: solves explicitly for y — requires all four prior points.)

Question 6

2019 ▪ Q6

[Figure: A graph showing a portion of the graph of f (a curve resembling a parabola opening upward, passing through $(0,3)$ with a minimum near $(1,1.8)$, rising steeply for $x>1$ and for $x<-1$) together with the line tangent to the graph of f at $x=0$ (a straight line through $(0,3)$ sloping downward to the right, passing near $(1.5,0)$), on a grid with x-axis from -2 to 2 and y-axis from 0 to 5 (approximately).]

| n | $f^{(n)}(0)$ | |—|—| | 2 | 3 | | 3 | $-\frac{23}{2}$ | | 4 | 54 |

A function f has derivatives of all orders for all real numbers x. A portion of the graph of f is shown above, along with the line tangent to the graph of f at $x=0$. Selected derivatives of f at $x=0$ are given in the table above.

(a) Write the third-degree Taylor polynomial for f about $x=0$.

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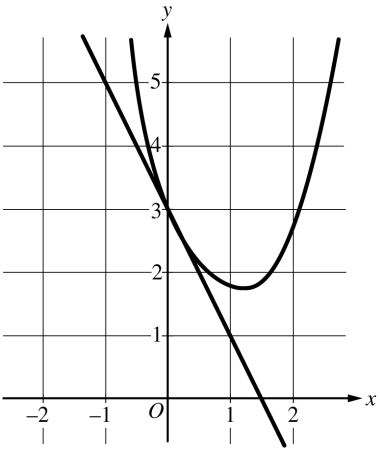
(b) Write the first three nonzero terms of the Maclaurin series for e^x. Write the second-degree Taylor polynomial for $e^x f(x)$ about $x = 0$.

(c) Let h be the function defined by $h(x) = \int_0^x f(t)\,dt$. Use the Taylor polynomial found in part (a) to find an approximation for $h(1)$.

(d) It is known that the Maclaurin series for h converges to $h(x)$ for all real numbers x. It is also known that the individual terms of the series for $h(1)$ alternate in sign and decrease in absolute value to 0. Use the alternating series error bound to show that the approximation found in part (c) differs from $h(1)$ by at most 0.45.

Question 6



n	$f^{(n)}(0)$
2	3
3	$-\frac{23}{2}$
4	54

Solution 6

(a) Mark scheme

- Given $f(0) = 3$ and $f'(0) = -2$ (and, from the problem, $f''(0) = 3$ and $f'''(0) = -23$), the third-degree Taylor polynomial for f about $x = 0$ is $3 - 2x + \frac{3}{2!}x^2 + \frac{-23}{3!}x^3 = 3 - 2x + \frac{3}{2}x^2 - \frac{23}{12}x^3$. Points: 1 for the first two terms $(3 - 2x)$, 1 for the remaining terms $(\frac{3}{2}x^2 - \frac{23}{12}x^3)$.

Solution 6

(b)

Mark scheme

- The first three nonzero terms of the Maclaurin series for e^x are $1 + x + \frac{1}{2!}x^2$. Using the Taylor polynomial for f from part (a) (truncated to degree 2, $f(x) \approx 3 - 2x + \frac{3}{2}x^2$), the second-degree Taylor polynomial for $e^x f(x)$ about $x = 0$ is $3\left(1 + x + \frac{1}{2!}x^2\right) - 2x(1+x) + \frac{3}{2}x^2(1) =$ $3 + (3-2)x + \left(\frac{3}{2} - 2 + \frac{3}{2}\right)x^2 = 3 + x + x^2$. Points: 1 for the three (nonzero) terms of the Maclaurin series for e^x, 1 for the three terms of the second-degree Taylor polynomial for $e^x f(x)$.

Solution 6

(c) Mark scheme

- $h(1) = \int_0^1 f(t)\,dt \approx \int_0^1 \left(3 - 2t + \frac{3}{2}t^2 - \frac{23}{12}t^3\right) dt = \left[3t - t^2 + \frac{1}{2}t^3 - \frac{23}{48}t^4\right]_0^1 = 3 - 1 + \frac{1}{2} - \frac{23}{48} = \frac{97}{48}$, using the degree-3 Taylor polynomial for f found in part (a) as an approximation for $f(t)$. Points: 1 for the antiderivative, 1 for the answer $97/48$.

Solution 6

(d) Mark scheme

- Because the terms of the Maclaurin series for $h(1)$ alternate in sign and decrease in absolute value to 0, the alternating series error bound says the error is at most the absolute value of the first omitted term. This first omitted term of $h(1)$'s series comes from the fourth-degree term of the Maclaurin series for f (coefficient $\frac{54}{4!}$, from $f''''(0) = 54$ given in the problem): $\int_0^1 \left(\frac{54}{4!}t^4\right) dt = \left[\frac{9}{20}t^5\right]_0^1 = \frac{9}{20}$. So the approximation from part (c) differs from $h(1)$ by at most $\left|\frac{9}{20}\right| = 0.45$. Points: 1 for using the fourth-degree term of the Maclaurin series for f, 1 for using the first omitted term of the series for $h(1)$, 1 for the error bound 0.45.

Question 6

2016 ▪ Q6

The function f has a Taylor series about $x = 1$ that converges to $f(x)$ for all x in the interval of convergence. It is known that $f(1) = 1$, $f'(1) = -\frac{1}{2}$, and the nth derivative of f at $x = 1$ is given by $f^{(n)}(1) = (-1)^n \frac{(n-1)!}{2^n}$ for $n \geq 2$.

(a) Write the first four nonzero terms and the general term of the Taylor series for f about $x = 1$.

(b) The Taylor series for f about $x = 1$ has a radius of convergence of 2. Find the interval of convergence. Show the work that leads to your answer.

(c) The Taylor series for f about $x = 1$ can be used to represent $f(1.2)$ as an alternating series. Use the first three nonzero terms of the alternating series to approximate $f(1.2)$.

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(d) Show that the approximation found in part (c) is within 0.001 of the exact value of $f(1.2)$.

Solution 6

(a) Mark scheme

- Using $f(1) = 1$, $f'(1) = -\frac{1}{2}$, and $f^{(n)}(1) = (-1)^n \frac{(n-1)!}{2^n}$ for $n \geq 2$ (so $f''(1) = \frac{1}{2^2} = \frac{1}{4}$, $f'''(1) = -\frac{2}{2^3} = -\frac{1}{4}$): $f(x) = 1 - \frac{1}{2}(x-1) + \frac{1}{8}(x-1)^2 - \frac{1}{24}(x-1)^3 + \cdots + \frac{(-1)^n}{n \cdot 2^n}(x-1)^n + \cdots$. Award 1 point for the first two terms (1 and $-\frac{1}{2}(x-1)$), 1 point for the third term $\frac{1}{8}(x-1)^2$, 1 point for the fourth term $-\frac{1}{24}(x-1)^3$, and 1 point for the correct general term $\frac{(-1)^n}{n \cdot 2^n}(x-1)^n$.

Solution 6

(b)

Mark scheme

- Radius of convergence $R = 2$ about center $x = 1$, so the series converges on the open interval $(-1, 3)$. At $x = -1$: the series becomes $1 + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$, which (up to overall sign) is the harmonic series and diverges. At $x = 3$: the series becomes $1 - 1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \cdots$, the alternating harmonic series, which converges. Therefore the interval of convergence is $-1 < x \leq 3$. Award 1 point for testing/identifying both endpoints, and 1 point for the correct analysis and final interval.

Solution 6

(c)

Mark scheme

- $f(1.2) \approx 1 - \frac{1}{2}(0.2) + \frac{1}{8}(0.2)^2 = 1 - 0.1 + 0.005 = 0.905$. Award 1 point for the correct approximation 0.905.

Solution 6

(d)

Mark scheme

- The series for $f(1.2)$ alternates with terms decreasing in magnitude to 0, so the Alternating Series Error Bound applies: $|f(1.2) - T_2(1.2)| \leq |\text{next term}| = \left|-\frac{1}{2^3 \cdot 3}(0.2)^3\right| = \frac{1}{3000} \approx 0.000333 \leq 0.001$. Award 1 point for citing the correct error-bound/next-term form, and 1 point for the numeric analysis showing it is ≤ 0.001.

Question 6

2015 ▪ Q6

The Maclaurin series for a function f is given by $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{n} x^n = x - \frac{3}{2}x^2 + 3x^3 - \cdots + \frac{(-3)^{n-1}}{n}x^n + \cdots$ and converges to $f(x)$ for $|x| < R$, where R is the radius of convergence of the Maclaurin series.

(a) Use the ratio test to find R.

(b) Write the first four nonzero terms of the Maclaurin series for f', the derivative of f. Express f' as a rational function for $|x| < R$.

(c) Write the first four nonzero terms of the Maclaurin series for e^x. Use the Maclaurin series for e^x to write the third-degree Taylor polynomial for $g(x) = e^x f(x)$ about $x = 0$.

Solution 6

(a) Mark scheme

- Let a_n be the nth term of the Maclaurin series. $\frac{a_{n+1}}{a_n} = \frac{(-3)^n x^{n+1}}{n+1} \cdot \frac{n}{(-3)^{n-1} x^n} = \frac{-3n}{n+1} x$. $\lim_{n\to\infty} \left| \frac{-3n}{n+1} x \right| = 3|x|$. $3|x| < 1 \Rightarrow |x| < \frac{1}{3}$. So the radius of convergence is $R = \frac{1}{3}$. Award 1 point for setting up the ratio $|a_{n+1}/a_n|$, 1 point for correctly computing the limit of the ratio $(3|x|)$, and 1 point for determining $R = 1/3$.

Solution 6

(b)

Mark scheme

- The first four nonzero terms of the Maclaurin series for f' are $1 - 3x + 9x^2 - 27x^3$ (term-by-term differentiation of the given series for f). As a rational function, $f'(x) = \frac{1}{1-(-3x)} = \frac{1}{1+3x}$ for $|x| < R$. Award 2 points for the correct first four nonzero terms, and 1 point for the correct rational function expression.

Solution 6

(c) Mark scheme

- The first four nonzero terms of the Maclaurin series for e^x are $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$. Multiplying by the Maclaurin series for f: $\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\left(x - \frac{3}{2}x^2 + 3x^3 - \cdots\right) = x - \frac{1}{2}x^2 + 2x^3 + \cdots$. So the third-degree Taylor polynomial for $g(x) = e^x f(x)$ about $x = 0$ is $T_3(x) = x - \frac{1}{2}x^2 + 2x^3$. Award 1 point for the correct first four nonzero terms of the Maclaurin series for e^x, and 2 points for the correct third-degree Taylor polynomial for g.