Past-paper walk-through

Alternating Series Error Bound ▪ 10.10

eXercise

Questions in this set

- 2026 Q6
- 2024 Q6
- 2022 Q6
- 2021 Q6
- 2019 Q6
- 2018 Q6
- 2017 Q6
- 2016 Q6

Reading the mark scheme

- **B1** a mark that stands on its own – the point is either made or not.
- **M1** a *method* mark: the right approach, even if the number is wrong.
- **A1** an *answer* mark, and it usually needs its M mark first.
- **C1** a working mark on the way to the answer.
- // separates answers the examiner accepts equally.
- **ecf** = error carried forward: a later part is marked on your own earlier answer.
- **owtte** = "or words to that effect" – the wording need not match.

Question 6

2026 ▪ Q6

The Maclaurin series for a function g is given by

$$\sum_{n=0}^{\infty} 2\left(\frac{-x}{5}\right)^n = 2 - \frac{2}{5}x + \frac{2}{25}x^2 - \frac{2}{125}x^3 + \frac{2}{625}x^4 - \cdots + 2\left(\frac{-x}{5}\right)^n + \cdots$$

and converges to $g(x)$ for $|x| < 5$.

(a) For $x > 0$, the Maclaurin series for g is an alternating geometric series. Find $g(3)$. Show the work that leads to your answer.

Question 6

2026 ▪ Q6

The Maclaurin series for a function g is given by

$$\sum_{n=0}^{\infty} 2\left(\frac{-x}{5}\right)^n = 2 - \frac{2}{5}x + \frac{2}{25}x^2 - \frac{2}{125}x^3 + \frac{2}{625}x^4 - \cdots + 2\left(\frac{-x}{5}\right)^n + \cdots$$

and converges to $g(x)$ for $|x| < 5$.

(b) The function f is defined by $f(x) = g'(x)$. Write the first four nonzero terms of the Maclaurin series for f.

Question 6

2026 ▪ Q6

The Maclaurin series for a function g is given by

$$\sum_{n=0}^{\infty} 2\left(\frac{-x}{5}\right)^n = 2 - \frac{2}{5}x + \frac{2}{25}x^2 - \frac{2}{125}x^3 + \frac{2}{625}x^4 - \cdots + 2\left(\frac{-x}{5}\right)^n + \cdots$$

and converges to $g(x)$ for $|x| < 5$.

(c) The second-degree Taylor polynomial for f about $x = 0$ is used to approximate $f\left(\frac{5}{2}\right)$ as $-\frac{3}{10}$, where f is the function defined in part B. Justify that $\left|f\left(\frac{5}{2}\right) - \left(-\frac{3}{10}\right)\right| \leq \frac{1}{5}$.

Question 6

2026 ▪ Q6

The Maclaurin series for a function g is given by

$$\sum_{n=0}^{\infty} 2\left(\frac{-x}{5}\right)^n = 2 - \frac{2}{5}x + \frac{2}{25}x^2 - \frac{2}{125}x^3 + \frac{2}{625}x^4 - \cdots + 2\left(\frac{-x}{5}\right)^n + \cdots$$

and converges to $g(x)$ for $|x| < 5$.

(d)(i) The function h is defined by $h(x) = 25g(x) - 2e^x$. Write the first three nonzero terms of the Maclaurin series for e^x.

(d)(ii) Write the first three nonzero terms of the Maclaurin series for h.

Question 6

2024 ▪ Q6

The Maclaurin series for a function f is given by $\sum_{n=1}^{\infty} \frac{(n+1)x^n}{n^2 6^n}$ and converges to $f(x)$ for all x in the interval of convergence. It can be shown that the Maclaurin series for f has a radius of convergence of 6.

(a) Determine whether the Maclaurin series for f converges or diverges at $x = 6$. Give a reason for your answer.

Question 6

2024 ▪ Q6

The Maclaurin series for a function f is given by $\sum_{n=1}^{\infty} \frac{(n+1)x^n}{n^2 6^n}$ and converges to $f(x)$ for all x in the interval of convergence. It can be shown that the Maclaurin series for f has a radius of convergence of 6.

(b) It can be shown that $f(-3) = \sum_{n=1}^{\infty} \frac{(n+1)(-3)^n}{n^2 6^n} = \sum_{n=1}^{\infty} \frac{n+1}{n^2} \left(-\frac{1}{2}\right)^n$ and that the first three terms of this series sum to $S_3 = -\frac{125}{144}$. Show that $|f(-3) - S_3| < \frac{1}{50}$.

Question 6

2024 ▪ Q6

The Maclaurin series for a function f is given by $\sum_{n=1}^{\infty} \frac{(n+1)x^n}{n^2 6^n}$ and converges to $f(x)$ for all x in the interval of convergence. It can be shown that the Maclaurin series for f has a radius of convergence of 6.

(c) Find the general term of the Maclaurin series for f', the derivative of f. Find the radius of convergence of the Maclaurin series for f'.

Question 6

2024 ▪ Q6

The Maclaurin series for a function f is given by $\sum_{n=1}^{\infty} \frac{(n+1)x^n}{n^2 6^n}$ and converges to $f(x)$ for all x in the interval of convergence. It can be shown that the Maclaurin series for f has a radius of convergence of 6.

(d) Let $g(x) = \sum_{n=1}^{\infty} \frac{(n+1)x^{2n}}{n^2 3^n}$. Use the ratio test to determine the radius of convergence of the Maclaurin series for g.

Solution 6

(a) Mark scheme

- The Maclaurin series for f is $\sum_{n=1}^{\infty} \frac{(n+1)x^n}{n^2 6^n}. At x = 6, thisbecomes \sum_{n=1}^{\infty} \frac{(n+1)6^n}{n^2 6^n} = \sum_{n=1}^{\infty} \frac{n+1}{n^2}. Because \frac{n+1}{n^2} > \frac{1}{n} foralln \geq 1 andtheharmonicseries \sum \frac{1}{n} diverges, theseries \sum \frac{n+1}{n^2} divergesbythecomparisontest(equivalently$

  $1). SotheMaclaurinseriesforfdivergesatx = 6. Points: 1forconsideringtheterm(n+1)6^n/(n^2 6^n)(oritssimplifiedform(n+1)/n^2)atx = 6, 1foracorrectcomparison - testorlimit - comparison - testargument(termsexceed1/n, orthelimitoftheratioto1/nequals1)correctlyconcludingdivergence$

Solution 6

(b) Mark scheme

- f(-
  3)=$\sum_{n=1}^{\infty} \frac{n+1}{n^2}\left(-\frac{1}{2}\right)^n isanalternatingserieswithtermsdecreasinginmagnitudeto0, andS_3 = -\frac{125}{144} isthesumofitsfirstthreeterms. Bythealternatingserieserrorbound, usingthefourthterm(n = 4) : \left|\frac{4+1}{4^2}\left(-\frac{1}{2}\right)^4\right| = \frac{5}{256} < \frac{5}{250} = \frac{1}{50}. Thus|f(-3) - S_3| < \frac{1}{50}. Points : 1forcorrectlyusingx = -3inthefourthtermoftheseries(i.e. theexpression(4+1)/4^2 \cdot (-1/2)^4, notjustitssimplifiedvalue5/256), 1forstating/usingthattheseriesisalternating(sothealte$
  $1/50.$

Solution 6

(c) Mark scheme

- Differentiating the Maclaurin series for f term by term, the general term of the Maclaurin series for f' is $\dfrac{(n+1)nx^{n-1}}{n^2 6^n} = \dfrac{(n+1)x^{n-1}}{n \cdot 6^n}. Becauseterm-by-$
$termdifferentiationofapowerseriesdoesnotchangeitsradiusofconvergence, andtheradiusofconverg$
$1 forthecorrectgeneralterm (n+1)x^{n-1}/(n \cdot 6^n) oranequivalentunsimplifiedform (e.g. (n+$
$1)nx^{n-1}/(n^2 6^n)), 1 forthecorrectradiusofconvergence 6, supported (thispointcanbeearnedindepen$
$testorroot - testcomputation).$

Solution 6

(d) Mark scheme

- For g(x)=$\sum_{n=1}^{\infty} \frac{(n+1)x^{2n}}{n^2 3^n}, applytheratiotest: \left|\frac{(n+2)x^{2n+2}/((n+1)^2 3^{n+1})}{(n+1)x^{2n}/(n^2 3^n)}\right| = \left|\frac{(n+2)x^{2n+2}}{(n+1)^2 3^{n+1}} \cdot \frac{n^2 3^n}{(n+1)x^{2n}}\right|. Takingthelimit: \lim_{n\to\infty} \left|\frac{(n+2)n^2}{(n+1)^3}\right| \cdot \left|\frac{x^2}{3}\right| = \left|\frac{x^2}{3}\right|. Setting \left|\frac{x^2}{3}\right| < 1 givesx^2 < 3, i.e. |x| < \sqrt{3}. SotheradiusofconvergenceoftheMaclaurinseriesforgis \sqrt{3}. Points: 1 forcorrectlysettinguptheratioofconsecutiveterms (withorwithoutabsolutevalues; onceearnedthis$

Question 6

2022 ▪ Q6

The function f is defined by the power series $f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + \frac{(-1)^n x^{2n+1}}{2n+1} + \cdots$ for all real numbers x for which the series converges.

(a) Using the ratio test, find the interval of convergence of the power series for f. Justify your answer.

(b) Show that $\left| f\left(\frac{1}{2}\right) - \frac{1}{2} \right| < \frac{1}{10}$. Justify your answer.

(c) Write the first four nonzero terms and the general term for an infinite series that represents $f'(x)$.

(d) Use the result from part (c) to find the value of $f'\left(\frac{1}{6}\right)$.

Solution 6

(a) Mark scheme

- Ratio test: $\lim_{n\to\infty}\left|\dfrac{(-1)^{n+1}x^{2n+3}/(2n+3)}{(-1)^n x^{2n+1}/(2n+1)}\right| = \lim_{n\to\infty}\left|x^2\dfrac{2n+1}{2n+3}\right| = |x^2| = x^2$. Need $x^2 < 1 \Rightarrow -1 < x < 1$ (interior of interval). At $x = -1$: series is $-1 + \frac{1}{3} - \frac{1}{5} + \cdots$, alternating with terms decreasing in absolute value to 0, converges by the Alternating Series Test. At $x = 1$: series is $1 - \frac{1}{3} + \frac{1}{5} - \cdots$, also alternating with terms decreasing to 0, converges by AST. So the interval of convergence is $-1 \leq x \leq 1$. 4 points: 1 for setting up the ratio, 1 for identifying the interior $-1 < x < 1$ from the limit, 1 for considering both endpoints ($x = \pm 1$), 1 for correct analysis at both endpoints and stating the full interval $-1 \leq x \leq 1$.

Solution 6

(b) Mark scheme

- $f(1/2)$ is an alternating series with first term $\frac{1}{2}$ and terms decreasing in absolute value to 0. By the alternating series error bound, $\left|f\left(\frac{1}{2}\right) - \frac{1}{2}\right| < |\text{second term}| = \left|\dfrac{(-1)^1(1/2)^3}{3}\right| = \dfrac{1}{24} < \dfrac{1}{10}$. 2 points: 1 for correctly using $x = \frac{1}{2}$ in the second term of the series, 1 for stating the series is alternating with terms decreasing to 0 and presenting the inequality Error $< \frac{1}{24} < \frac{1}{10}$ (stating Error $= 1/24$ alone does not earn this point).

Solution 6

(c)

Mark scheme

- $f'(x) = 1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n} + \cdots$ (term-by-term derivative of the series for f). 2 points: 1 for presenting the first four nonzero terms $(1, -x^2, x^4, -x^6)$, 1 for the correct general term $(-1)^n x^{2n}$.

Solution 6

(d) Mark scheme

- $f(1/6) = 1 - (1/6)^2 + (1/6)^4 - (1/6)^6 + \cdots$ is a geometric series with first term $a = 1$ and common ratio $r = -1/36$.
  $f(1/6) = \dfrac{a}{1-r} = \dfrac{1}{1-(-1/36)} = \dfrac{1}{37/36} = \dfrac{36}{37}$. 1 point for the correct answer $36/37$, contingent on correctly identifying the series from part (c) as geometric.

Question 6

2021 ▪ Q6

The function g has derivatives of all orders for all real numbers. The Maclaurin series for g is given by $g(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2e^n + 3}$ on its interval of convergence.

(a) State the conditions necessary to use the integral test to determine convergence of the series $\sum_{n=0}^{\infty} \frac{1}{e^n}$.

Use the integral test to show that $\sum_{n=0}^{\infty} \frac{1}{e^n}$ converges.

Question 6

2021 ▪ Q6

The function g has derivatives of all orders for all real numbers. The Maclaurin series for g is given by $g(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2e^n + 3}$ on its interval of convergence.

(b) Use the limit comparison test with the series $\sum_{n=0}^{\infty} \frac{1}{e^n}$ to show that the series $g(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2e^n + 3}$ converges absolutely.

Question 6

2021 ▪ Q6

The function g has derivatives of all orders for all real numbers. The Maclaurin series for g is given by $g(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2e^n + 3}$ on its interval of convergence.

(c) Determine the radius of convergence of the Maclaurin series for g.

Question 6

2021 ▪ Q6

The function g has derivatives of all orders for all real numbers. The Maclaurin series for g is given by $g(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2e^n + 3}$ on its interval of convergence.

(d) The first two terms of the series $g(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2e^n + 3}$ are used to approximate $g(1)$.

Use the alternating series error bound to determine an upper bound on the error of the approximation.

Solution 6

(a) Mark scheme

- Conditions for the integral test: e^{-x} must be positive, decreasing, and continuous on $[0, \infty)$ — all hold. $\int_0^\infty e^{-x}\,dx = \lim_{b\to\infty}\int_0^b e^{-x}\,dx = \lim_{b\to\infty}(-e^{-b} + e^0) = 0 + 1 = 1$, which is finite, so the improper integral converges. Therefore, by the integral test, $\sum_{n=0}^{\infty} \frac{1}{e^n}$ converges. (1 pt: states all three conditions — positive, decreasing, continuous; 1 pt: correctly writes the improper integral or an equivalent limit; 1 pt: correctly evaluates the limit using proper limit notation, finds e^0 or 1, and concludes convergence — a response that only uses a geometric-series approach earns 0 points here.)

Solution 6

(b) Mark scheme

- $\lim_{n\to\infty} \dfrac{1/e^n}{|(-1)^n/(2e^n+3)|} = \lim_{n\to\infty} \dfrac{2e^n+3}{e^n} = 2$. This limit exists and is positive, so by the limit comparison test (using absolute values), since $\sum_{n=0}^{\infty} \frac{1}{e^n}$ converges, $\sum_{n=0}^{\infty} \left|\frac{(-1)^n}{2e^n+3}\right|$ also converges. Therefore $g(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2e^n+3}$ converges absolutely. (1 pt: correctly sets up the limit comparison with limit notation, with or without absolute values explicit at setup; 1 pt: correctly evaluates the limit as a finite positive number and concludes absolute convergence specifically of $g(1)$ — requires absolute-value reasoning to have appeared.)

Solution 6

(c) Mark scheme

- Ratio test: $\left|\dfrac{a_{n+1}x^{n+1}}{a_n x^n}\right| = \dfrac{2e^n+3}{2e^{n+1}+3}\,|x| \to \dfrac{1}{e}|x|$ as $n \to \infty$. Setting $\dfrac{1}{e}|x| < 1$ gives $|x| < e$. The radius of convergence is $R = e$. (1 pt: correctly sets up the ratio of consecutive terms; 1 pt: correctly computes the limit of the ratio as $|x|/e$ [finite, nonzero coefficient]; 1 pt: explicit final answer $R = e$ from solving the inequality — presenting only an interval like $-e < x < e$ without identifying R does not earn this point.)

Solution 6

(d) Mark scheme

- The terms of the alternating series $g(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2e^n+3}$ decrease in magnitude to 0, so the alternating series error bound applies: the error from using the first two terms ($n = 0, 1$) to approximate $g(1)$ is bounded by the absolute value of the next (third, $n = 2$) term: Error $\leq \left| \dfrac{(-1)^2}{2e^2+3} \right| = \dfrac{1}{2e^2+3}$. (1 pt: correct upper bound $\frac{1}{2e^2+3}$.)

Question 6

2019 ▪ Q6

[Figure: A graph showing a portion of the graph of f (a curve resembling a parabola opening upward, passing through $(0,3)$ with a minimum near $(1,1.8)$, rising steeply for $x>1$ and for $x<-1$) together with the line tangent to the graph of f at $x=0$ (a straight line through $(0,3)$ sloping downward to the right, passing near $(1.5,0)$), on a grid with x-axis from -2 to 2 and y-axis from 0 to 5 (approximately).]

n	$f^{(n)}(0)$
2	3
3	$-\frac{23}{2}$
4	54

A function f has derivatives of all orders for all real numbers x. A portion of the graph of f is shown above, along with the line tangent to the graph of f at $x=0$. Selected derivatives of f at $x=0$ are given in the table above.

(a) Write the third-degree Taylor polynomial for f about $x=0$.

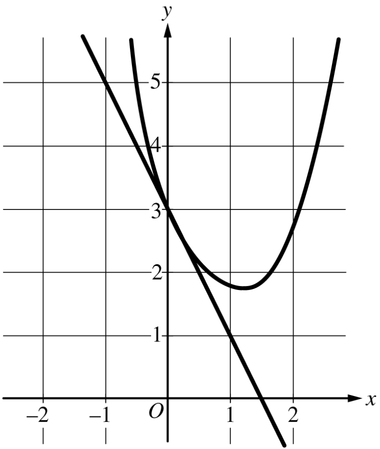
Question 6

(b) Write the first three nonzero terms of the Maclaurin series for e^x. Write the second-degree Taylor polynomial for $e^x f(x)$ about $x = 0$.

(c) Let h be the function defined by $h(x) = \int_0^x f(t)\,dt$. Use the Taylor polynomial found in part (a) to find an approximation for $h(1)$.

(d) It is known that the Maclaurin series for h converges to $h(x)$ for all real numbers x. It is also known that the individual terms of the series for $h(1)$ alternate in sign and decrease in absolute value to 0. Use the alternating series error bound to show that the approximation found in part (c) differs from $h(1)$ by at most 0.45.

Question 6



n	$f^{(n)}(0)$
2	3
3	$-\frac{23}{2}$
4	54

Solution 6

(a) Mark scheme

- Given $f(0) = 3$ and $f'(0) = -2$ (and, from the problem, $f''(0) = 3$ and $f'''(0) = -23$), the third-degree Taylor polynomial for f about $x = 0$ is $3 - 2x + \frac{3}{2!}x^2 + \frac{-23}{3!}x^3 = 3 - 2x + \frac{3}{2}x^2 - \frac{23}{12}x^3$. Points: 1 for the first two terms $(3 - 2x)$, 1 for the remaining terms $(\frac{3}{2}x^2 - \frac{23}{12}x^3)$.

Solution 6

(b)

Mark scheme

- The first three nonzero terms of the Maclaurin series for e^x are $1 + x + \frac{1}{2!}x^2$. Using the Taylor polynomial for f from part (a) (truncated to degree 2, $f(x) \approx 3 - 2x + \frac{3}{2}x^2$), the second-degree Taylor polynomial for $e^x f(x)$ about $x = 0$ is $3\left(1 + x + \frac{1}{2!}x^2\right) - 2x(1 + x) + \frac{3}{2}x^2(1) = 3 + (3 - 2)x + \left(\frac{3}{2} - 2 + \frac{3}{2}\right)x^2 = 3 + x + x^2$. Points: 1 for the three (nonzero) terms of the Maclaurin series for e^x, 1 for the three terms of the second-degree Taylor polynomial for $e^x f(x)$.

Solution 6

(c) Mark scheme

- $h(1) = \int_0^1 f(t)\,dt \approx \int_0^1 \left(3 - 2t + \frac{3}{2}t^2 - \frac{23}{12}t^3\right) dt = \left[3t - t^2 + \frac{1}{2}t^3 - \frac{23}{48}t^4\right]_0^1 = 3 - 1 + \frac{1}{2} - \frac{23}{48} = \frac{97}{48}$, using the degree-3 Taylor polynomial for f found in part (a) as an approximation for $f(t)$. Points: 1 for the antiderivative, 1 for the answer $97/48$.

Solution 6

(d) Mark scheme

- Because the terms of the Maclaurin series for $h(1)$ alternate in sign and decrease in absolute value to 0, the alternating series error bound says the error is at most the absolute value of the first omitted term. This first omitted term of $h(1)$'s series comes from the fourth-degree term of the Maclaurin series for f (coefficient $\frac{54}{4!}$, from $f'''(0) = 54$ given in the problem): $\int_0^1 \left(\frac{54}{4!}t^4\right) dt = \left[\frac{9}{20}t^5\right]_0^1 = \frac{9}{20}$. So the approximation from part (c) differs from $h(1)$ by at most $\left|\frac{9}{20}\right| = 0.45$. Points: 1 for using the fourth-degree term of the Maclaurin series for f, 1 for using the first omitted term of the series for $h(1)$, 1 for the error bound 0.45.

Question 6

2018 ▪ Q6

The Maclaurin series for $\ln(1+x)$ is given by

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n+1}\frac{x^n}{n} + \cdots .$$

On its interval of convergence, this series converges to $\ln(1+x)$. Let f be the function defined by

$$f(x) = x\ln\left(1 + \frac{x}{3}\right).$$

(a) Write the first four nonzero terms and the general term of the Maclaurin series for f.

Question 6

(b) Determine the interval of convergence of the Maclaurin series for f. Show the work that leads to your answer.

(c) Let $P_4(x)$ be the fourth-degree Taylor polynomial for f about $x = 0$. Use the alternating series error bound to find an upper bound for $|P_4(2) - f(2)|$.

Solution 6

(a) Mark scheme

- The first four nonzero terms of the Maclaurin series for f are $\frac{x^2}{3} - \frac{x^3}{2 \cdot 3^2} + \frac{x^4}{3 \cdot 3^3} - \frac{x^5}{4 \cdot 3^4}$. The general term is $(-1)^{n+1}\frac{x^{n+1}}{n \cdot 3^n}$. Points: 1 for the correct first four nonzero terms, 1 for the correct general term.

Solution 6

(b) Mark scheme

- Using the ratio test: $\lim_{n\to\infty}\left|\frac{(-1)^{n+2}x^{n+2}/((n+1)3^{n+1})}{(-1)^{n+1}x^{n+1}/(n\cdot 3^n)}\right| = \lim_{n\to\infty}\left|\frac{-x}{3}\cdot\frac{n}{n+1}\right| = \left|\frac{x}{3}\right|$. This is < 1 for $|x| < 3$, so the radius of convergence is 3. (Equivalently, using the known radius of convergence 1 for the Maclaurin series of $\ln(1+x)$ and $f(x) = x\ln(1+x/3)$, substitute $x/3$ to also get radius 3.) Checking the endpoints: at $x = -3$, the series is $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(-3)^{n+1}}{n\cdot 3^n} = \sum_{n=1}^{\infty}\frac{3}{n}$, which diverges by comparison to the harmonic series. At $x = 3$, the series is $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{3^{n+1}}{n\cdot 3^n} = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{3}{n}$, which converges by the alternating series test. The interval of convergence of the

Solution 6

Maclaurin series for f is $-3 < x \leq 3$. Points: 1 for setting up the ratio test, 1 for correctly computing the limit of the ratio, 1 for the correct radius of convergence (3), 1 for considering both endpoints $x = \pm 3$, 1 for the correct analysis and interval of convergence ($-3 < x \leq 3$).

Solution 6

(c)

Mark scheme

- By the alternating series error bound, an upper bound for $|P_4(2) - f(2)|$ is the magnitude of the next (fifth-degree) term of the alternating series: $|P_4(2) - f(2)| < \left|-\frac{2^5}{4 \cdot 3^4}\right| = \frac{8}{81}$. Points: 1 for using the fifth-degree term as the error bound, 1 for the correct answer $(8/81)$.

Question 6

2017 ▪ Q6

$$f(0) = 0$$
$$f'(0) = 1$$
$$f^{(n+1)}(0) = -n \cdot f^{(n)}(0) \quad \text{for all } n \geq 1$$

A function f has derivatives of all orders for $-1 < x < 1$. The derivatives of f satisfy the conditions above. The Maclaurin series for f converges to $f(x)$ for $|x| < 1$.

(a) Show that the first four nonzero terms of the Maclaurin series for f are $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$, and write the general term of the Maclaurin series for f.

Question 6

2017 ▪ Q6

$$f(0) = 0$$
$$f'(0) = 1$$
$$f^{(n+1)}(0) = -n \cdot f^{(n)}(0) \quad \text{for all } n \geq 1$$

A function f has derivatives of all orders for $-1 < x < 1$. The derivatives of f satisfy the conditions above. The Maclaurin series for f converges to $f(x)$ for $|x| < 1$.

(b) Determine whether the Maclaurin series described in part (a) converges absolutely, converges conditionally, or diverges at $x = 1$. Explain your reasoning.

Question 6

2017 ▪ Q6

$$f(0) = 0$$
$$f'(0) = 1$$
$$f^{(n+1)}(0) = -n \cdot f^{(n)}(0) \quad \text{for all } n \geq 1$$

A function f has derivatives of all orders for $-1 < x < 1$. The derivatives of f satisfy the conditions above. The Maclaurin series for f converges to $f(x)$ for $|x| < 1$.

(c) Write the first four nonzero terms and the general term of the Maclaurin series for $g(x) = \int_0^x f(t)\,dt$.

Question 6

2017 ▪ Q6

$$f(0) = 0$$
$$f'(0) = 1$$
$$f^{(n+1)}(0) = -n \cdot f^{(n)}(0) \quad \text{for all } n \geq 1$$

A function f has derivatives of all orders for $-1 < x < 1$. The derivatives of f satisfy the conditions above. The Maclaurin series for f converges to $f(x)$ for $|x| < 1$.

(d) Let $P_n\left(\frac{1}{2}\right)$ represent the nth-degree Taylor polynomial for g about $x = 0$ evaluated at $x = \frac{1}{2}$, where g is the function defined in part (c). Use the alternating series error bound to show that

Question 6

$$\left| P_4\left(\frac{1}{2}\right) - g\left(\frac{1}{2}\right) \right| < \frac{1}{500}.$$

Solution 6

(a) Mark scheme

- $f(0) = 0$, $f'(0) = 1$, $f''(0) = -1(1) = -1$, $f'''(0) = -2(-1) = 2$, $f^{(4)}(0) = -3(2) = -6$ (from the given recursive derivative relation). The first four nonzero terms of the Maclaurin series: $0 + 1x + \frac{-1}{2!}x^2 + \frac{2}{3!}x^3 + \frac{-6}{4!}x^4 = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$. General term: $\frac{(-1)^{n+1}x^n}{n}$. 1 pt for correctly computing $f''(0)$, $f'''(0)$, $f^{(4)}(0)$, 1 pt for verifying the four given terms match, 1 pt for the correct general term $(-1)^{n+1}x^n/n$.

Solution 6

(b) Mark scheme

- For $x = 1$ the Maclaurin series becomes $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ (the alternating harmonic series). It does not converge absolutely because the harmonic series $\sum 1/n$ diverges. The terms alternate in sign and decrease in magnitude to 0, so by the Alternating Series Test the series converges conditionally. 2 pts for the correct conclusion (converges conditionally) with both parts of the reasoning (harmonic series diverges $\Rightarrow$ not absolute convergence; alternating $+$ decreasing to 0 $\Rightarrow$ converges).

Solution 6

(c) Mark scheme

- $g(x) = \int_0^x f(t)\,dt = \int_0^x \left(t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots + \frac{(-1)^{n+1} t^n}{n} + \cdots \right) dt =$ $\left[\frac{t^2}{2} - \frac{t^3}{3\cdot 2} + \frac{t^4}{4\cdot 3} - \frac{t^5}{5\cdot 4} + \cdots + \frac{(-1)^{n+1} t^{n+1}}{(n+1)n} + \cdots \right]_0^x =$ $\frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{12} - \frac{x^5}{20} + \cdots + \frac{(-1)^{n+1} x^{n+1}}{(n+1)n} + \cdots$. 1 pt for the first two terms $(x^2/2 - x^3/6)$, 1 pt for the remaining shown terms $(x^4/12 - x^5/20)$, 1 pt for the correct general term $(-1)^{n+1}x^{n+1}/[(n+1)n]$.

Solution 6

(d)

Mark scheme

- The terms of the series for $g(1/2)$ alternate in sign and decrease in magnitude to 0, so by the Alternating Series Error Bound, $\left|P_4(\frac{1}{2}) - g(\frac{1}{2})\right|$ is bounded by the magnitude of the first unused (5th) term: $\left|-\frac{(1/2)^5}{20}\right|$. Thus $\left|P_4(\frac{1}{2}) - g(\frac{1}{2})\right| \leq \left|-\frac{(1/2)^5}{20}\right| = \frac{1}{32 \cdot 20} = \frac{1}{640} < \frac{1}{500}$. 1 pt for correctly applying the alternating series error bound to establish the inequality.

Question 6

2016 ▪ Q6

The function f has a Taylor series about $x = 1$ that converges to $f(x)$ for all x in the interval of convergence. It is known that $f(1) = 1$, $f'(1) = -\frac{1}{2}$, and the nth derivative of f at $x = 1$ is given by $f^{(n)}(1) = (-1)^n \frac{(n-1)!}{2^n}$ for $n \geq 2$.

(a) Write the first four nonzero terms and the general term of the Taylor series for f about $x = 1$.

(b) The Taylor series for f about $x = 1$ has a radius of convergence of 2. Find the interval of convergence. Show the work that leads to your answer.

(c) The Taylor series for f about $x = 1$ can be used to represent $f(1.2)$ as an alternating series. Use the first three nonzero terms of the alternating series to approximate $f(1.2)$.

Question 6

(d) Show that the approximation found in part (c) is within 0.001 of the exact value of $f(1.2)$.

Solution 6

(a) Mark scheme

- Using $f(1) = 1$, $f'(1) = -\dfrac{1}{2}$, and $f^{(n)}(1) = (-1)^n \dfrac{(n-1)!}{2^n}$ for $n \geq 2$ (so $f''(1) = \dfrac{1}{2^2} = \dfrac{1}{4}$, $f'''(1) = -\dfrac{2}{2^3} = -\dfrac{1}{4}$): $f(x) = 1 - \dfrac{1}{2}(x-1) + \dfrac{1}{8}(x-1)^2 - \dfrac{1}{24}(x-1)^3 + \cdots + \dfrac{(-1)^n}{n \cdot 2^n}(x-1)^n + \cdots$. Award 1 point for the first two terms (1 and $-\frac{1}{2}(x-1)$), 1 point for the third term $\frac{1}{8}(x-1)^2$, 1 point for the fourth term $-\frac{1}{24}(x-1)^3$, and 1 point for the correct general term $\frac{(-1)^n}{n \cdot 2^n}(x-1)^n$.

Solution 6

(b)

Mark scheme

- Radius of convergence $R = 2$ about center $x = 1$, so the series converges on the open interval $(-1, 3)$. At $x = -1$: the series becomes $1 + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$, which (up to overall sign) is the harmonic series and diverges. At $x = 3$: the series becomes $1 - 1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \cdots$, the alternating harmonic series, which converges. Therefore the interval of convergence is $-1 < x \leq 3$. Award 1 point for testing/identifying both endpoints, and 1 point for the correct analysis and final interval.

Solution 6

(c)

Mark scheme

- $f(1.2) \approx 1 - \frac{1}{2}(0.2) + \frac{1}{8}(0.2)^2 = 1 - 0.1 + 0.005 = 0.905$. Award 1 point for the correct approximation 0.905.

Solution 6

(d)

Mark scheme

- The series for $f(1.2)$ alternates with terms decreasing in magnitude to 0, so the Alternating Series Error Bound applies: $|f(1.2) - T_2(1.2)| \leq |\text{next term}| = \left|-\frac{1}{2^3 \cdot 3}(0.2)^3\right| = \frac{1}{3000} \approx 0.000333 \leq 0.001$. Award 1 point for citing the correct error-bound/next-term form, and 1 point for the numeric analysis showing it is ≤ 0.001.