Applications of Integration

AP Calculus AB ▪ Unit 8

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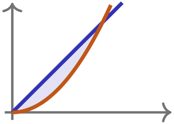
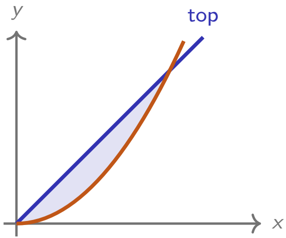


What we will cover

1. Average value of a function
2. Motion with integrals
3. Net change in context
4. Area between curves
5. Volumes by cross section
6. Volumes of revolution



1 Average value

Average value of a function

The **average value** 平均值 over $[a, b]$:

$$f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

It is the constant height of a rectangle on $[a, b]$ with the same area as under f.

> Do not confuse it with the average rate of change (which divides the *change in f* by the interval).

Average value of $f(x) = x^2$ on $[0, 3]$

$$\frac{1}{3}\int_0^3 x^2 dx = \frac{1}{3}\left[\frac{x^3}{3}\right]_0^3 = 3.$$

Position, Velocity, and Acceleration Using Integrals

Integration reverses the motion links of Unit 4.

- **Displacement 位移** (net change in position) $= \displaystyle\int_{t_1}^{t_2} v(t)\,dt$.
- **Total distance travelled 总路程** $= \displaystyle\int_{t_1}^{t_2}\ dt$ – integrate **speed**, so direction changes add up instead of cancelling.
- **Position** at a later time $= x(t_1) + \displaystyle\int_{t_1}^{t} v(s)\,ds$; **velocity** from acceleration $= v(t_1) + \int a$.

2

Motion

Motion, run backwards

Integration reverses Unit 4's links:

- **displacement** 位移 $= \displaystyle\int_{t_1}^{t_2} v(t)\,dt$
- **total distance** 总路程 $= \displaystyle\int_{t_1}^{t_2} |v(t)|\,dt$
- position $= x(t_1) + \displaystyle\int_{t_1}^{t} v(s)\,ds$

> The displacement-vs-distance distinction (that absolute value) is a classic graded point.

Net change in an applied context

The **net change** 净变化 theorem is the everyday tool:

$$\text{final} = \text{initial} + \int_a^b (\text{rate})\, dt.$$

"How much water is in the tank at $t = 5$"
$= \text{start} + \int_0^5 (\text{in} - \text{out})\, dt.$

> Watch the signs (in minus out) and the units.

Many multi-part FRQs are built entirely on this idea – often paired with "write, but do not evaluate," an integral expression.

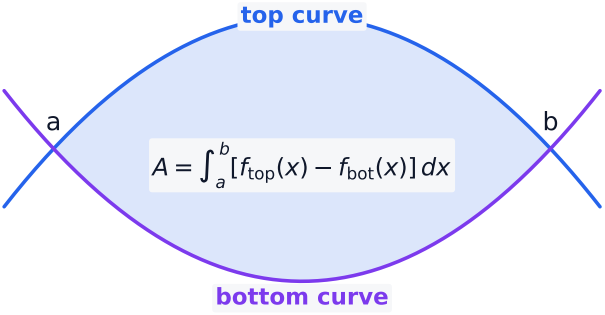
3 Area between curves

Area between two curves

On $[a, b]$ with f on top and g below:

$$A = \int_a^b \left[f(x) - g(x) \right] dx.$$

> Find the intersections first to set the limits, and always subtract bottom from top.



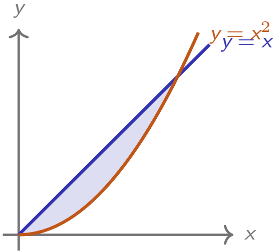
Worked example – the area between $y = x$ and $y = x^2$

Find the enclosed area

They cross at $x = 0$ and $x = 1$; on $(0, 1)$ the line is on top:

$$A = \int_0^1 (x - x^2)\, dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$$

$$= \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$



Horizontal slices, and more than two crossings

Slice in y

$$A = \int_c^d \left[x_{\text{right}}(y) - x_{\text{left}}(y)\right] dy$$

Choose x- or y-slices to keep one top/bottom pair over the whole region.

Several crossings

$$A = \int_a^b \left|f(x) - g(x)\right| dx$$

or split into a sum at each crossing – decide which curve is higher on each piece.

4 Volumes

Volumes with known cross sections

If a solid has a known **cross section** 横截面 perpendicular to an axis, the volume is the integral of its area:

$$V = \int_a^b A(x)\,dx.$$

> The exam phrase "the base of a solid… cross sections are squares" signals exactly this.

square of side s: $A = s^2$

rectangle, height ks: $A = k\,s^2$

equilateral **triangle**: $A = \frac{\sqrt{3}}{4}s^2$

semicircle, diameter s: $A = \frac{\pi}{8}s^2$

> Set the shape's key length equal to the slice length $s(x) = f(x) - g(x)$, then integrate.

Volumes of revolution – the disc method

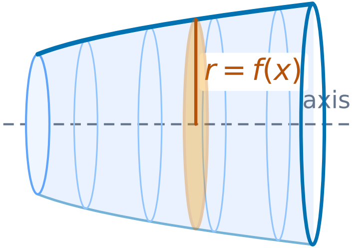
Revolving a region about an axis makes a **solid of revolution** 旋转体. If the region touches the axis, each slice is a **disc** 圆盘:

$$V = \pi \int_a^b \left[r(x)\right]^2 dx.$$

About the y-axis: express x as a function of y and integrate in y.

Volumes of revolution – the disc method, in detail

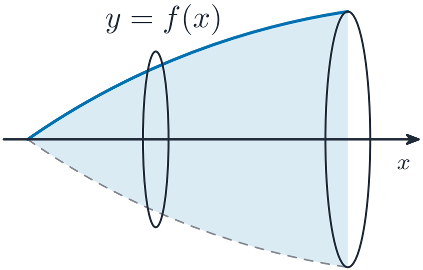
rotate $y = f(x)$ about the x-axis



$$V = \pi \int_a^b [f(x)]^2 \, dx \quad \text{(disk method about the } x\text{-axis)}$$

Revolving a region about an axis makes a **solid of revolution** 旋转体. Where the region touches the axis, every slice is a **disc** 圆盘.

Volume with the Disc Method (About the x- or y-Axis)



rotate the region around the x-axis to sweep a solid

> Rotating a region about an axis sweeps out a solid of revolution

Volumes with Cross Sections: Squares and Rectangles

A potter shaping a vessel: solids of revolution are volumes formed by rotating a plane region

Exam tips

- Area between curves is $\int(\text{top} - \text{bottom})\,dx$ – find the intersection points for the limits and keep top minus bottom.
- For a **volume of revolution**, add up disc/washer cross-sections of area πr^2 (or $\pi(R^2 - r^2)$).
- The **average value** of f on $[a, b]$ is $\frac{1}{b-a}\int_a^b f\,dx$.
- Accumulated change is $\int$ of a rate: total $=$ initial value $+ \int_a^b(\text{rate})\,dt$.
- Integration means "adding up infinitely many tiny pieces" – set up the integrand as one thin slice.

5 Recap

Unit 8 – key takeaways

1 **Average value**

$\frac{1}{b-a}\int_a^b f$ – not the average rate of change

2 **Motion**

displacement integrates v; distance integrates $|v|$

3 **Net change**

final $=$ initial $+ \int$ rate – watch signs and units

4 **Area & volume**

top minus bottom; $V = \int A(x)\,dx$; discs use πr^2

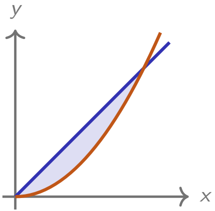
Check yourself

1. Which integral gives total distance, $\int v$ or $\int |v|$?

2. Find the area between $y = x$ and $y = x^2$.

3. Write the average value of f on $[a, b]$.

4. A solid's cross sections are squares of side $s(x)$. Give V.



Answers **1.** $\int |v|$ **2.** $\frac{1}{6}$ **3.** $\frac{1}{b-a}\int_a^b f$ **4.** $V = \int_a^b [s(x)]^2 dx$