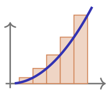
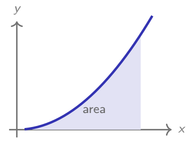
Integration & Accumulation of Change

AP Calculus AB ▪ Unit 6

AP Calculus AB



What we will cover



1 Accumulation

Accumulation of change

Differentiation found rates. Integration runs it in reverse: given a rate, find the **accumulated change** 累积变化.

> The area between a rate graph and the x-axis is the total accumulation.

Water tank

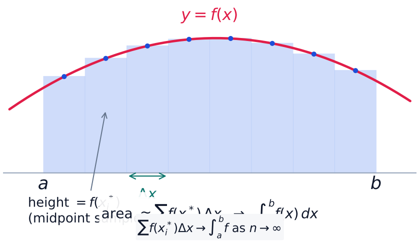
2 Riemann sums

Approximating area with a Riemann sum

Split the interval and add rectangle areas – a **Riemann sum** 黎曼和. Each rectangle is $f(x)\,\Delta x$.

> As the strips narrow, the sum approaches the definite integral.

Approximating area with a Riemann sum, in detail



> Split the interval and add rectangle areas – a **Riemann sum** 黎曼和.

The four standard estimates

Left sum – height from each strip's left endpoint	**Right** sum – height from the right endpoint
Midpoint sum – height from the **midpoint** 中点	**Trapezoidal** sum 梯形法 – average the two endpoint heights

> Strips may be uniform or nonuniform – read the widths from the table.

Over- or underestimate?

Judge from the behaviour of the function:

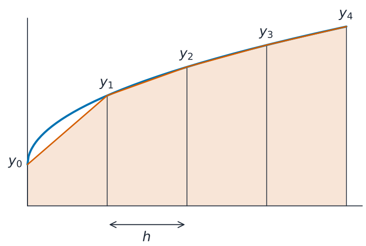
- increasing f: left sum under, right sum over
- concave up: trapezoidal over
- concave down: trapezoidal under

Finding an area without an antiderivative, and its units

geometry 几何	simple regions – triangles and rectangles – can be found by area formula alone
a left or right sum	rectangles at one end of each subinterval
a midpoint sum	rectangles at the centre – usually the most accurate of the three
a trapezoidal sum 梯形和	the average of the left and right sums; it overestimates a concave-up function
the units	the area's unit is the rate's unit times the input's unit
the example	a rate in vehicles per hour, integrated over hours, gives vehicles

> Stating the units of a definite integral is a marked step on the free-response paper, and it is also the fastest check that you integrated the right thing.

Over- or underestimate?, in detail



> Exam parts ask you to state which and why.

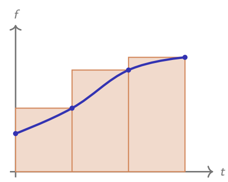
Worked example – a right sum from a table

$f(0) = 3,\ f(2) = 5,\ f(4) = 8,\ f(6) = 9.$
Estimate $\int_0^6 f$ with a right sum, $n = 3$.

$\Delta x = 2$; use each strip's right endpoint:

$$2\big(f(2) + f(4) + f(6)\big) = 2(5 + 8 + 9) = 44.$$

f is increasing, so this is an overestimate; the left sum $2(3 + 5 + 8) = 32$ would be an underestimate.



From Riemann sum to integral notation

As the widths shrink to zero the sum approaches an exact value – the **definite integral** 定积分:

$$\int_a^b f(x)\,dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*)\,\Delta x_i.$$

> A definite integral is the limit of a Riemann sum – be ready to translate each into the other.

3 The Fundamental Theorem

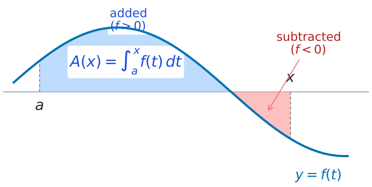
Accumulation functions and the FTC

A definite integral with a variable upper limit is an **accumulation function** 累积函数. The **Fundamental Theorem** 微积分基本定理 says differentiation undoes it:

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

> So $g(x) = \int_a^x f$ has $g' = f$ and $g'' = f'$.

$A'(x) = f(x)$ (Fundamental Theorem)



Behaviour of an accumulation function

Because $g'(x) = f(x)$, everything from Unit 5 applies – read it off the graph of f:

g **increasing** where $f > 0$, **decreasing** where $f < 0$	g has a **local extremum** where f crosses zero
g **concave up** where f is increasing	a **value** of g is the signed area up to x

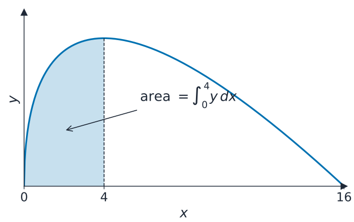
Properties of definite integrals

$$\int_a^a f = 0 \qquad \int_b^a f = -\int_a^b f \qquad \int_a^b kf = k\int_a^b f$$

$$\int_a^b (f \pm g) = \int_a^b f \pm \int_a^b g \qquad \int_a^c f + \int_c^b f = \int_a^b f$$

> The last one – splitting at an interior point – lets you build a total from pieces read off a graph.

The Fundamental Theorem and Evaluating Integrals



> A definite integral is the signed area between the curve and the x-axis

The theorem, and one algebraic tool

Part 1 — The **fundamental theorem of calculus**: if $F(x) = \int_a^x f(t)\,dt$ then $F'(x) = f(x)$ – differentiation undoes accumulation.

Part 2 — $\int_a^b f(x)\,dx = F(b) - F(a)$ for any antiderivative F – which is what makes a definite integral computable.

With a chain rule — If the upper limit is $g(x)$, then the derivative is $f(g(x))\,g'(x)$.

Polynomial long division — **Polynomial long division** first when the integrand is an improper rational function – the quotient integrates directly, and only the remainder needs more work.

> The two parts say the same thing twice: accumulation and rate are inverse operations.

Evaluating an integral – the second part of the FTC

If $F' = f$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Find any **antiderivative** 原函数, then subtract its values at the limits.

> Net change: $\int_a^b g'(t)\,dt = g(b) - g(a)$, so $g(5) = g(0) + \int_0^5 g'$.

Evaluate $\int_1^3 (2x+1)\,dx$

An antiderivative is $F(x) = x^2 + x$:

$F(3) - F(1) = (9+3) - (1+1) = 10$.

4 Antiderivatives

Antiderivatives and indefinite integrals

An **indefinite integral** 不定积分 is the family of all antiderivatives:

$$\int f(x)\,dx = F(x) + C.$$

> Never drop the **constant of integration** 积分常数.

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C\ (n \neq -1)$$

$$\int \frac{1}{x} dx = \ln|x| + C$$

$$\int e^x dx = e^x + C$$

$$\int \cos x\,dx = \sin x + C$$

u-substitution – reversing the chain rule

Choose an inside function $u = g(x)$, so $du = g'(x)\,dx$:

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

> Look for a function and its derivative both present.

Evaluate $\int 2x\cos(x^2)\,dx$

Inside is $u = x^2$, and $du = 2x\,dx$ is already there:

$$\int \cos u\,du = \sin u + C = \sin(x^2) + C.$$

Spotting that $2x$ is exactly $\frac{du}{dx}$ is the whole trick.

For a definite integral: change the limits to u-values, or convert back to x first.

Selecting Techniques for Antidifferentiation

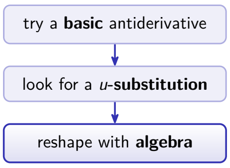
A skill topic: match the integral to a method.

> Try a basic antiderivative first; look for a u-substitution (an inside function whose derivative is also present); use algebra (division, completing the square.

Reshaping an integrand, and choosing a technique

Two algebraic set-up moves:

- **long division** 多项式长除法 when the top degree $\geq$ the bottom
- **completing the square** 配方法 to reach an arctangent or log form



Naming the structure first prevents wasted effort.

Exam tips

- Integration is antidifferentiation; use the **power rule** $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$ and don't forget the $+C$.
- The **Fundamental Theorem** links the two: $\int_a^b f'(x)\,dx = f(b) - f(a)$, and $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$.
- Approximate a definite integral with **Riemann sums** or the trapezoidal rule from a table of values.
- A definite integral is a signed area (below the axis counts negative); split at sign changes for total area.
- Use **u-substitution** and remember to change the limits (or back-substitute) accordingly.

5 Recap

Unit 6 – key takeaways

1 Accumulation
signed area under a rate graph is the total change

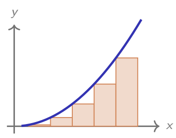
2 Riemann sums
left, right, midpoint, trapezoid – and say over or under

3 The FTC
$\frac{d}{dx}\int_a^x f = f(x)$ and $\int_a^b f = F(b) - F(a)$

4 u-substitution
spot a function and its derivative; never drop $+C$

Check yourself

1. For an increasing f, is a left sum an over- or underestimate?
2. What is $\frac{d}{dx}\int_a^x f(t)\,dt$?
3. Evaluate $\int_1^3 (2x+1)\,dx$.
4. In $\int 2x\cos(x^2)dx$, what is u?



left sum

Answers **1.** underestimate **2.** $f(x)$ **3.** 10 **4.** $u = x^2$