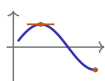


Analytical Applications of Differentiation

AP Calculus AB • Unit 5

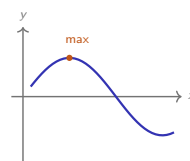
AP Calculus AB



Analytical Applications of Differentiation

What we will cover

- 1 The Mean Value Theorem
- 2 Critical points & extrema
- 3 The first derivative test
- 4 Concavity & inflection
- 5 Connecting f , f' and f''
- 6 Optimization



1 Mean Value Theorem

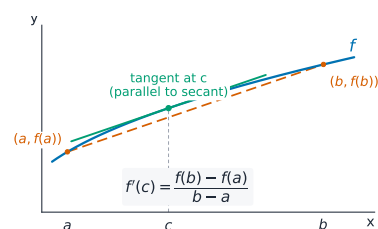
Analytical Applications of Differentiation

Mean Value Theorem

If f is continuous on $[a, b]$ and differentiable on (a, b) , then for some c in (a, b) :

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Somewhere inside, the instantaneous rate equals the average rate – some tangent is parallel to the secant.



Analytical Applications of Differentiation

Mean Value Theorem

Exam skill – justifying the MVT

The **Mean Value Theorem** 中值定理 is an **existence** theorem, so the marks are in the justification:

- 1 state f is continuous on $[a, b]$ and differentiable on (a, b)
- 2 compute $\frac{f(b) - f(a)}{b - a}$
- 3 conclude “by the MVT there is a c in (a, b) with $f'(c)$ equal to that value”

Both hypotheses must be named – dropping one loses the mark even with the right c .

Verify the MVT for $f(x) = x^2$ on $[1, 3]$

A polynomial, so continuous and differentiable everywhere. Average rate $\frac{9 - 1}{2} = 4$; solve $f'(c) = 2c = 4$ to get $c = 2$, which lies in $(1, 3)$.

2 Critical points

Extreme values and critical points

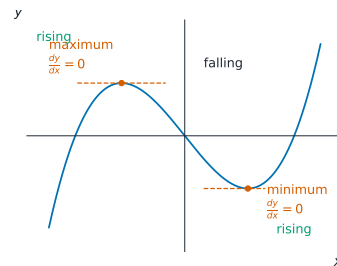
The Extreme Value

Theorem 极值定理: a function **continuous** on a **closed** interval attains **both** an absolute max and an absolute min. A **critical point** 临界点 is an interior point where $f'(x) = 0$ or f' does not exist.



Garden fence

At a maximum or minimum the derivative is zero



At a maximum or minimum the derivative is zero

Increasing and decreasing intervals

The **first derivative** says where f rises or falls:

- $f'(x) > 0 \Rightarrow f$ is **increasing** 递增
- $f'(x) < 0 \Rightarrow f$ is **decreasing** 递减



Mountain extrema

The first derivative test

Classify a critical point c by how f' **changes sign** there:

f' turns $+$ $\rightarrow$ $-$
local maximum
极大值

f' turns $-$ $\rightarrow$ $+$
local minimum
极小值

f' does **not** change sign
neither

Always state the **sign change** as your justification.

Sketching a Function and Its Derivatives

Key features of f , f' , and f'' mirror each other.

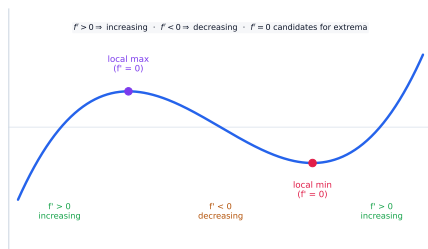
- f increasing $\Leftrightarrow f'$ above the axis; f has a local max $\Leftrightarrow f'$ crosses from $+$ to $-$.
- f concave up $\Leftrightarrow f''$ increasing $\Leftrightarrow f''$ above the axis.

Worked example – classify the critical points

$$f(x) = x^3 - 3x^2$$

$f'(x) = 3x^2 - 6x = 3x(x - 2)$, zero at $x = 0, 2$. Signs: $f' > 0$ for $x < 0$; $f' < 0$ on $(0, 2)$; $f' > 0$ for $x > 2$. $+$ $\rightarrow$ $-$ at $x = 0$: **local max**, $f(0) = 0$. $- \rightarrow +$ at $x = 2$: **local min**, $f(2) = -4$.

Worked example – classify the critical points, in detail



$$f(x) = x^3 - 3x^2 \quad f'(x) = 3x^2 - 6x = 3x(x - 2), \text{ zero at } x = 0, 2.$$

Analytical Applications of Differentiation

Critical points

The candidates test – absolute extrema

On a **closed** interval, absolute extrema occur only at **critical points** or **endpoints**:

- 1 list every critical point in $[a, b]$ and both endpoints
- 2 evaluate f at each candidate
- 3 largest is the absolute max; smallest the absolute min

Show the **table of values** – the comparison is the argument.

endpoints a, b
+
every critical point
= the full candidate list

Analytical Applications of Differentiation

Critical points

Extrema: local, global, and how to justify

Local (relative) extrema

Local (relative) extrema are the highest or lowest value in some neighbourhood – and by Fermat's theorem they occur only at critical points.

Absolute extrema

On a closed interval these are found by the candidates test: evaluate at every critical point *and* both endpoints.

Justifying

The first-derivative test names the sign change of f' ; the second-derivative test uses the sign of f'' at the critical point.

What earns the mark

State the value *and* the reason – “ f changes from positive to negative at $x = 2$, so $f(2)$ is a local maximum”.

A critical point is a candidate, never a conclusion. The justification is the marked part of an AP answer.

3 Concavity

Analytical Applications of Differentiation

Concavity

Concavity and points of inflection

The **second derivative** describes bending:

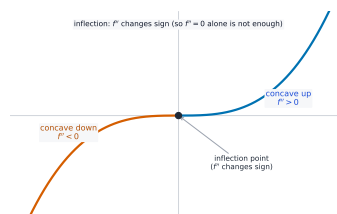
- $f'' > 0$: **concave up** 上凹 (f increasing)
- $f'' < 0$: **concave down** 下凹 (f decreasing)

A **point of inflection** 拐点 is where f'' **changes sign** – not merely where $f'' = 0$.

Analytical Applications of Differentiation

Concavity

Concavity and points of inflection, in detail



Concavity comes from the sign of the second derivative; it flips at an inflection point

The second derivative test

An alternative at a critical point c with $f'(c) = 0$:

$f''(c) > 0 \Rightarrow$ **local minimum**

$f''(c) < 0 \Rightarrow$ **local maximum**

$f''(c) = 0 \Rightarrow$ **inconclusive**

If $f''(c) = 0$, fall back on the **first derivative test**.

Special case: if a continuous function has only **one** critical point on an interval and it is a local extremum, it is the **absolute** extremum there.

Connecting f , f' and f''

A very common setup: you are given the graph of f' and asked about f .

f' **increasing** $\Leftrightarrow f'$ **above the axis**

f has a **local max** $\Leftrightarrow f'$ crosses $+$ $\rightarrow$ $-$

f **concave up** $\Leftrightarrow f'$ **increasing**

f has an **inflection** $\Leftrightarrow f'$ has a local extremum

Answer questions about f using the **height** and **slope** of the f' graph.

4 Optimization

Solving an optimization problem

Optimization 最优化 is the same machinery aimed at a real goal:

- 1 write the quantity as a function of **one** variable (use the constraint to eliminate the rest)
- 2 state the allowed interval
- 3 find and **test** the critical points
- 4 answer with **units** and interpretation

constraint $\rightarrow$ one variable

differentiate, set $= 0$

test, then interpret

Worked example – the farmer's fence

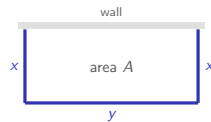
100 m of fence, three sides (a wall is the fourth). Maximize the area.

Ends x , side y : constraint $2x + y = 100$, so $y = 100 - 2x$.

$$A(x) = x(100 - 2x) = 100x - 2x^2$$

$$A'(x) = 100 - 4x = 0 \Rightarrow x = 25$$

$A'' = -4 < 0$, a maximum: $y = 50$ and $A = 1250 \text{ m}^2$.



Behaviour of implicit relations

It all extends to **implicitly** defined relations:

- horizontal tangent where $\frac{dy}{dx} = 0$
- vertical tangent where $\frac{dy}{dx}$ is **undefined**

$\frac{dy}{dx}$ usually involves **both** x and y – substitute it back in when finding $\frac{d^2y}{dx^2}$, then read concavity from the sign.

5

Recap

Unit 5 – key takeaways

1 MVT

some tangent is parallel to the secant – name both hypotheses

2 Critical points

$f' = 0$ or undefined: candidates only, so test each

3 The two tests

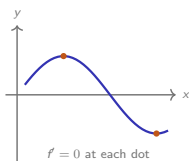
f' sign change, or $f' > 0$ min / $f' < 0$ max

4 Justify

cite the sign of f' or f'' – the reasoning earns the marks

Check yourself

- 1 f' turns $-$ to $+$ at c . What is c ?
- 2 $f'(c) = 0$ at a critical point. What does the test tell you?
- 3 Where can an absolute max on $[a, b]$ occur?
- 4 Is every critical point an extremum?



Answers 1. a local minimum 2. inconclusive – use the f' test 3. a critical point or an endpoint 4. no