Analytical Applications of Differentiation

AP Calculus AB ▪ Unit 5

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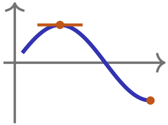
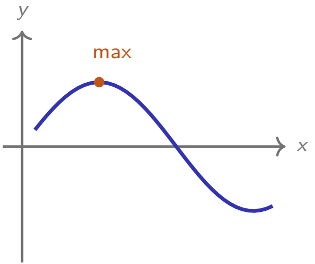


What we will cover

1. The Mean Value Theorem
2. Critical points & extrema
3. The first derivative test
4. Concavity & inflection
5. Connecting f f' and f''
6. Optimization



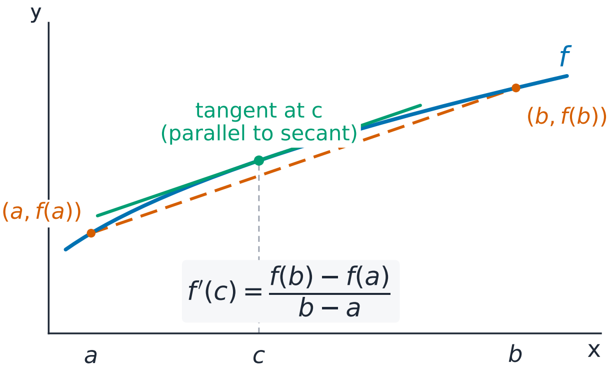
1 Mean Value Theorem

The Mean Value Theorem

If f is continuous on $[a, b]$ and differentiable on (a, b), then for some c in (a, b):

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

> Somewhere inside, the instantaneous rate equals the average rate – some tangent is parallel to the secant.



Exam skill – justifying the MVT

The **Mean Value Theorem** 中值定理 is an existence theorem, so the marks are in the justification:

1. state f is continuous on $[a, b]$ and differentiable on (a, b)
2. compute $\dfrac{f(b) - f(a)}{b - a}$
3. conclude "by the MVT there is a c in (a, b) with $f'(c)$ equal to that value"

Both hypotheses must be named – dropping one loses the mark even with the right c.

Verify the MVT for $f(x) = x^2$ on $[1, 3]$

A polynomial, so continuous and differentiable everywhere. Average rate $\dfrac{9-1}{2} = 4$; solve $f'(c) = 2c = 4$ to get $c = 2$, which lies in $(1, 3)$.

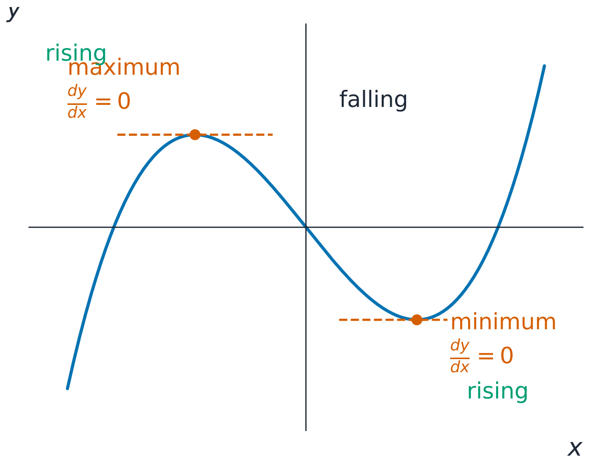
2 Critical points

Extreme values and critical points

The **Extreme Value Theorem** 极值定理: a function continuous on a closed interval attains both an absolute max and an absolute min. A **critical point** 临界点 is an interior point where $f'(x) = 0$ or f' does not exist.

Garden fence

At a maximum or minimum the derivative is zero



At a maximum or minimum the derivative is zero

Increasing and decreasing intervals

The first derivative says where f rises or falls:

- $f'(x) > 0 \;\Rightarrow\;$ f is **increasing** 递增
- $f'(x) < 0 \;\Rightarrow\;$ f is **decreasing** 递减

Mountain extrema

The first derivative test

Classify a critical point c by how f' changes sign there:

f' turns $+ \to -$	f' turns $- \to +$	f' does **not** change sign
local maximum 极大值	**local minimum** 极小值	**neither**

Always state the sign change as your justification.

Sketching a Function and Its Derivatives

Key features of f, f', and f'' mirror each other.

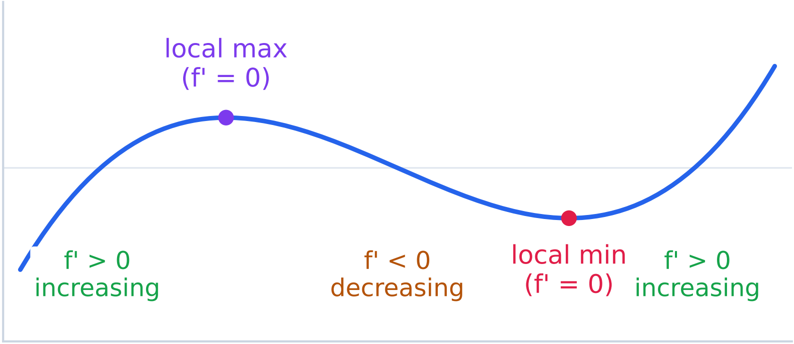
- f increasing $\Leftrightarrow$ f' above the axis; f has a local max $\Leftrightarrow$ f' crosses from $+$ to $-$.
- f concave up $\Leftrightarrow$ f' increasing $\Leftrightarrow$ f'' above the axis.

Worked example – classify the critical points

$f(x) = x^3 - 3x^2$

$f'(x) = 3x^2 - 6x = 3x(x-2)$, zero at $x = 0, 2$. Signs: $f' > 0$ for $x < 0$; $f' < 0$ on $(0, 2)$; $f' > 0$ for $x > 2$. $+ \to -$ at $x = 0$: local max, $f(0) = 0$.
$- \to +$ at $x = 2$: local min, $f(2) = -4$.

Worked example – classify the critical points, in detail



$f' > 0 \Rightarrow$ increasing · $f' < 0 \Rightarrow$ decreasing · $f' = 0$ candidates for extrema

> $f(x) = x^3 - 3x^2$ $f'(x) = 3x^2 - 6x = 3x(x-2)$, zero at $x = 0,\ 2$.

The candidates test – absolute extrema

On a closed interval, absolute extrema occur only at critical points or endpoints:

1. list every critical point in $[a, b]$ and both endpoints
2. evaluate f at each candidate
3. largest is the absolute max; smallest the absolute min

> Show the table of values – the comparison *is* the argument.

endpoints a, b

$+$

every critical point

$=$ the full candidate list

Extrema: local, global, and how to justify

Local (relative) extrema	**Local (relative) extrema** are the highest or lowest value in some neighbourhood – and by Fermat's theorem they occur only at critical points.
Absolute extrema	On a closed interval these are found by the candidates test: evaluate at every critical point *and* both endpoints.
Justifying	The first-derivative test names the sign change of f'; the second-derivative test uses the sign of f'' at the critical point.
What earns the mark	State the value *and* the reason – "f' changes from positive to negative at $x = 2$, so $f(2)$ is a local maximum".

A critical point is a candidate, never a conclusion. The justification is the marked part of an AP answer.

3 Concavity

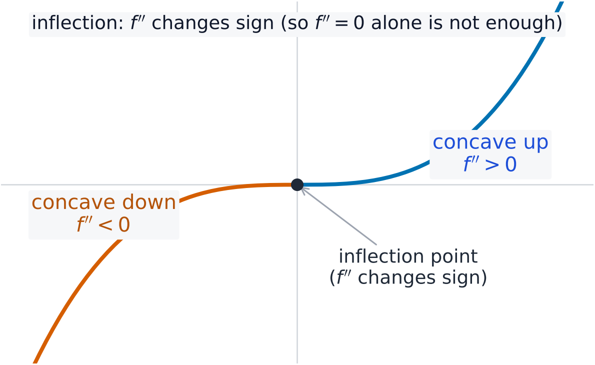
Concavity and points of inflection

The second derivative describes bending:

- $f'' > 0$: **concave up** 上凹 (f' increasing)
- $f'' < 0$: **concave down** 下凹 (f' decreasing)

> A **point of inflection** 拐点 is where f'' changes sign – not merely where $f'' = 0$.

Concavity and points of inflection, in detail



Concavity comes from the sign of the second derivative; it flips at an inflection point

The second derivative test

An alternative at a critical point c with $f'(c) = 0$:

$f''(c) > 0 \Rightarrow$ **local minimum**

$f''(c) < 0 \Rightarrow$ **local maximum**

$f''(c) = 0 \Rightarrow$ **inconclusive**

If $f''(c) = 0$, fall back on the first derivative test.

Special case: if a continuous function has only one critical point on an interval and it is a local extremum, it is the absolute extremum there.

Connecting f, f' and f''

A very common setup: you are given the graph of f' and asked about f.

f **increasing** $\Leftrightarrow$ f' above the axis

f has a **local max** $\Leftrightarrow$ f' crosses $+ \;\rightarrow\; -$

f **concave up** $\Leftrightarrow$ f' increasing

f has an **inflection** $\Leftrightarrow$ f' has a local extremum

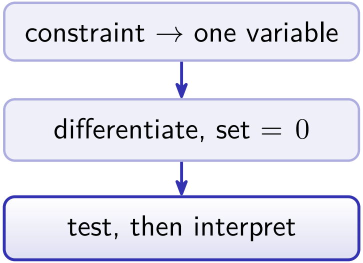
> Answer questions about f using the height and slope of the f' graph.

4 Optimization

Solving an optimization problem

Optimization 最优化 is the same machinery aimed at a real goal:

1. write the quantity as a function of one variable (use the constraint to eliminate the rest)
2. state the allowed interval
3. find and test the critical points
4. answer with units and interpretation



Worked example – the farmer's fence

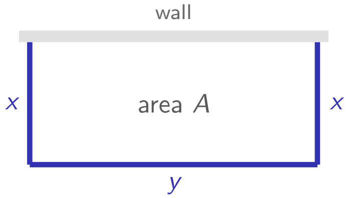
100 m of fence, three sides (a wall is the fourth). Maximize the area.

Ends x, side y: constraint $2x + y = 100$, so $y = 100 - 2x$.

$$A(x) = x(100 - 2x) = 100x - 2x^2$$

$$A'(x) = 100 - 4x = 0 \;\Rightarrow\; x = 25$$

$A'' = -4 < 0$, a maximum: $y = 50$ and $A = 1250\ \mathrm{m}^2$.



Behaviour of implicit relations

It all extends to implicitly defined relations:

- horizontal tangent where $\dfrac{dy}{dx} = 0$
- vertical tangent where $\dfrac{dy}{dx}$ is undefined

> $\dfrac{dy}{dx}$ usually involves both x and y – substitute it back in when finding $\dfrac{d^2y}{dx^2}$, then read concavity from the sign.

5 Recap

Unit 5 – key takeaways

1 MVT

some tangent is parallel to the secant – name both hypotheses

2 Critical points

$f' = 0$ or undefined: candidates only, so test each

3 The two tests

f' sign change, or $f'' > 0$ min / $f'' < 0$ max

4 Justify

cite the sign of f' or f'' – the reasoning earns the marks

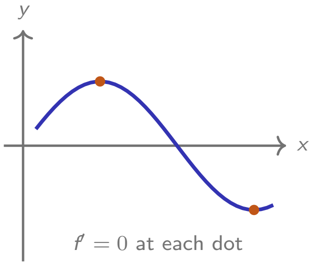
Check yourself

1. f' turns $-$ to $+$ at c. What is c?

2. $f'(c) = 0$ at a critical point. What does the test tell you?

3. Where can an absolute max on $[a, b]$ occur?

4. Is every critical point an extremum?



Answers **1.** a local minimum **2.** inconclusive – use the f' test **3.** a critical point or an endpoint **4.** no