Chain Rule, Implicit & Inverse

AP Calculus AB ▪ Unit 3

AP Calculus AB

$x \to g \xrightarrow{u} f \to y$

What we will cover

1 Composite functions
2 The chain rule
3 Implicit differentiation
4 Derivatives of inverse functions
5 Inverse trig derivatives
6 Higher-order derivatives & strategy

$x \xrightarrow{u = g(x)} u \xrightarrow{y = f(u)} y$

1 Chain rule

What makes a function composite?

A **composite function** 复合函数 $f(g(x))$ nests one function inside another.

> Always ask two questions: what is the inside, and what is its derivative?

Chain rule matryoshka

The chain rule

Differentiate outer-then-inner:

$$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x).$$

In Leibniz notation, with $y = f(u)$ and $u = g(x)$:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

> The inner derivative $g'(x)$ is the most-forgotten factor on the exam.

differentiate the **outer**, leave the in-

multiply by the derivative of the **inside**

$\cos(x^2) \cdot 2x$

Worked example – a power of a polynomial

Differentiate $h(x) = (2x^2 + 1)^5$

Outer is $(\,\cdot\,)^5$, inner is $2x^2 + 1$:

$$h'(x) = 5(2x^2+1)^4 \cdot \frac{d}{dx}(2x^2+1)$$

$$= 5(2x^2+1)^4 \cdot 4x = 20x(2x^2+1)^4.$$

outer $(\,\cdot\,)^5 \to 5(\,\cdot\,)^4$

inner $2x^2 + 1 \to 4x$

Exam skill – the chain rule from a table

A table gives values; no formula is needed. With $h(x) = f(g(x))$:

$$h'(a) = f'(g(a)) \cdot g'(a).$$

Read $g(a)$ first, then look up f' at that value – not at a.

Find $h'(1)$

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	6	4	2	5
2	3	7	1	9

$g(1) = 2$, so
$h'(1) = f'(2) \cdot g'(1) = 7 \times 5 = 35$.

2 Implicit

Implicit differentiation

Some curves are defined **implicitly** 隐式地 – an equation in x and y not solved for y, e.g. $x^2 + y^2 = 25$.

1. differentiate both sides in x
2. each y-term picks up a $\frac{dy}{dx}$ (chain rule)
3. solve algebraically for $\frac{dy}{dx}$

$$x^2 + y^2 = 25$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

Worked example – tangent to a circle

Tangent to $x^2 + y^2 = 25$ at $(3, 4)$

From $\frac{dy}{dx} = -\frac{x}{y}$, at $(3, 4)$:

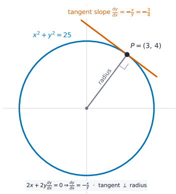
$$\frac{dy}{dx} = -\frac{3}{4}.$$

So the tangent is

$$y - 4 = -\tfrac{3}{4}(x - 3).$$

The tangent is perpendicular to the radius – a free sanity check.

Worked example – tangent to a circle, in detail



Implicit differentiation of $x^2 + y^2 = 25$ gives $\frac{dy}{dx} = -\frac{x}{y}$, so at $(3, 4)$ the gradient is $-\frac{3}{4}$ – perpendicular to the radius.

Exam skill – "Show that $\frac{dy}{dx} = \ldots$"

The target is given, so the marks are in the algebra, not the answer.

1. differentiate both sides
2. product/chain rule on mixed xy terms
3. collect every $\frac{dy}{dx}$ on one side
4. factor, then divide

A correct final line that skips the algebra earns little.

Horizontal tangent
$\frac{dy}{dx} = 0$: numerator $= 0$

Vertical tangent
denominator $= 0$

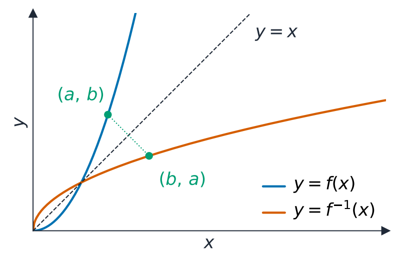
3 Inverse functions

Derivatives of inverse functions

If g is the **inverse function** 反函数 of f (so $f(g(x)) = x$):

$$g'(x) = \frac{1}{f'(g(x))}.$$

Slopes at matching points are **reciprocals** 倒数 – the graphs are mirror images in $y = x$.



Worked example – inverse from a point

$f(2) = 5$ and $f'(2) = 3$; g is the inverse of f

$(2, 5)$ lies on f, so $(5, 2)$ lies on g. Then

$$g'(5) = \frac{1}{f'(2)} = \frac{1}{3}.$$

Swap the coordinates first, then take the reciprocal of f' at the original x.

on f: $(2, 5)$ $\xrightarrow{\text{swap}}$ on g: $(5, 2)$

Inverse trigonometric derivatives

The same idea gives the **inverse trigonometric functions** 反三角函数 you must know:

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\arctan x = \frac{1}{1+x^2} \qquad \frac{d}{dx}\operatorname{arcsec} x = \frac{1}{|x|\sqrt{x^2-1}}$$

Chain them when the input is a function: $\frac{d}{dx}\arctan(3x) = \frac{3}{1+9x^2}$.

4 Higher-order & strategy

Higher-order derivatives

Differentiating f' gives the **second derivative** 二阶导数 f'':

$$f''(x) = \frac{d^2y}{dx^2} = y'', \qquad f^{(n)}(x) = \frac{d^ny}{dx^n}.$$

f'' measures how the slope is changing – it drives **concavity** 凹凸性 (Unit 5) and acceleration (Unit 4).

$s(t)$ position → $s'(t)$ velocity → $s''(t)$ acceleration

Choosing the right rule

Read the structure from the outside in: name the outermost operation, apply its rule, recurse inward.

a **sum**? differentiate term by term	a **product** / **quotient**? that rule (chain inside)
a **composite**? chain rule	given **implicitly**? implicit differentiation

> Neatness prevents the sign and bookkeeping errors that cost the marks.

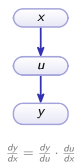
5 Recap

Unit 3 – key takeaways

1. **Chain rule**
   $f'(g(x)) \cdot g'(x)$ – never drop the inner derivative
2. **Implicit**
   differentiate both sides; each y gives a dy/dx; then solve
3. **Inverse**
   $g'(x) = 1/f'(g(x))$ – reciprocal slopes at swapped points
4. **Second derivative**
   differentiate twice; velocity → acceleration

Check yourself

1. Differentiate $\cos(3x)$.
2. For $x^2 + y^2 = 25$, what is $\frac{dy}{dx}$ at $(3, 4)$?
3. $f(2) = 5$, $f'(2) = 3$. If $g = f^{-1}$, what is $g'(5)$?
4. Which factor do students most often forget?



> **Answers** **1.** $-3\sin(3x)$ **2.** $-\frac{3}{4}$ **3.** $\frac{1}{3}$ **4.** the inner derivative $g'(x)$