Differentiation: Definition & Properties

AP Calculus AB ▪ Unit 2

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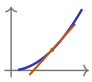
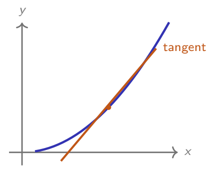


What we will cover



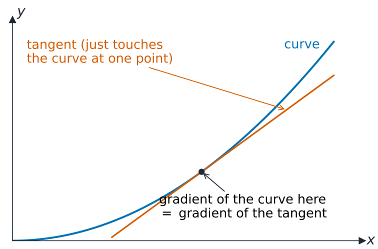
1 The derivative

Average vs instantaneous rate

Over an interval, the average rate is a **difference quotient** 差商:

$$\frac{f(a+h)-f(a)}{h}=\frac{f(x)-f(a)}{x-a}.$$

At a point, the **instantaneous** 瞬时 rate is the gradient of the **tangent** 切线.



From secant to tangent

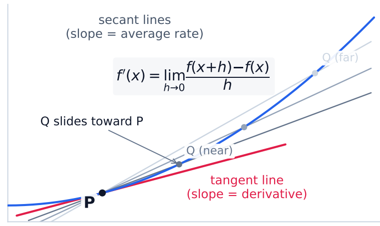
Let the point a vary and the derivative becomes a **new function**:

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}.$$

This is the **definition of the derivative** (sometimes called differentiating "by first...

Its value at each x is the instantaneous rate of change there.

From secant to tangent, in detail



As Q slides toward P, each secant's slope (an average rate) approaches the tangent's slope – the derivative $f'(a)$.

The derivative as a function, and its notation

Let a vary and the derivative becomes a new function:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

This is differentiating **by first principles** 用定义求导.

$\frac{dy}{dx}$ $f'(x)$ y'

three names for the same thing

Be ready to move between graph, table, formula and words.

The tangent line – a routine exam task

The tangent at $x = a$ passes through $(a, f(a))$ with slope $f'(a)$:

$$y - f(a) = f'(a)\,(x - a).$$

Keep point-slope form ready – you need a point and a slope, nothing else.

$f(x) = x^2$ at $x = 3$

Point: $f(3) = 9$. Slope: $f'(x) = 2x$, so $f'(3) = 6$.

$$y - 9 = 6(x - 3).$$

Exam skill – estimating f' from a table

With no formula, use a difference quotient over a small interval around a:

$$f'(a) \approx \frac{f(b) - f(c)}{b - c}, \quad c < a < b.$$

Full credit needs the numbers and the **units** 单位 – output per input.

"Approximate $M'(7.5)$ over $5 \le t \le 10$"

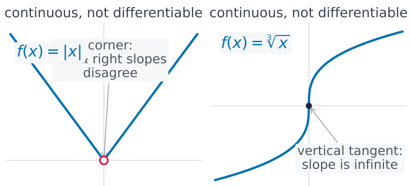
$$M'(7.5) \approx \frac{M(10) - M(5)}{10 - 5}$$

Values on both sides of 7.5 give the best estimate. Answer in units of M per unit of t.

2 When a derivative exists

Differentiable ⇒ continuous (not the reverse)

If f is **differentiable** 可导 at a point, then f is **continuous** 连续 there. A continuous function can still fail to be differentiable:



f differentiable at a ⇒ f continuous at a; converse fails at corners and vertical tangents

Use the contrapositive: if f is not continuous at a, it is not differentiable at a.

The notations, and the tangent line

the notations 记号	$\frac{dy}{dx}$, $f'(x)$, y' – all naming the same object
which to use	Leibniz $\frac{dy}{dx}$ when the variables matter; Lagrange $f'(x)$ when the function does
the second derivative	$\frac{d^2y}{dx^2}$ or $f''(x)$
the tangent line	give it as $y - f(a) = f'(a)(x - a)$
why that form	it needs no rearrangement and no risk of an arithmetic slip in the intercept
the check	substitute $x = a$: the equation must give $y = f(a)$

Writing the tangent in point–slope form is a marked convention, not a preference. Converting it to $y = mx + c$ adds a step and an opportunity to lose a mark.

Differentiability is the stronger condition

the relationship	differentiable $\Rightarrow$ continuous
why	a derivative is a limit that cannot exist where the function jumps
the converse is false	a continuous function can fail to be differentiable
a corner	$\lvert x\rvert$ at $x = 0$ – the left and right slopes differ
a cusp	$x^{2/3}$ at $x = 0$ – the slopes go to $+\infty$ and $-\infty$
a vertical tangent	$x^{1/3}$ at $x = 0$ – the slope is infinite, so no finite derivative

On a justification question, say which direction of the implication you are using. "It is continuous, therefore differentiable" is the error the question is set to catch.

3 Rules

The power rule

The **power rule** 幂法则 handles any real power:

$$\frac{d}{dx} x^r = r\,x^{r-1}.$$

Rewrite as a power first: $\frac{1}{x} = x^{-1}$, $\sqrt{x} = x^{1/2}$.

$x^5 \;\to\; 5x^4$

$\frac{1}{x} = x^{-1} \;\to\; -x^{-2}$

$\sqrt{x} = x^{1/2} \;\to\; \frac{1}{2}x^{-1/2}$

Constant, multiple, sum & difference

Constant
$\frac{d}{dx}k = 0$

Constant multiple
$\frac{d}{dx}(kf) = kf'$

Sum / difference
$(f \pm g)' = f' \pm g'$

Differentiate a polynomial term by term

$$\frac{d}{dx}\left(4x^3 - 5x + 7\right) = 12x^2 - 5.$$

The constant 7 contributes nothing – it does not change.

Four building blocks to know by heart

$\frac{d}{dx}\sin x = \cos x$	$\frac{d}{dx}\cos x = -\sin x$	$\frac{d}{dx}e^x = e^x$	$\frac{d}{dx}\ln x = \frac{1}{x}$

Note the minus sign on $\cos$, and that e^x is its own derivative.

A limit that is really a derivative

If a limit has the shape of the definition

$$\lim_{h\to 0}\frac{f(a+h) - f(a)}{h},$$

do not expand it – just evaluate $f'(a)$.

Evaluate the limit

$$\lim_{h\to 0}\frac{\sin\left(\frac{\pi}{2} + h\right) - 1}{h}$$

This is $\left.\frac{d}{dx}\sin x\right|_{x=\pi/2} = \cos\frac{\pi}{2} = 0.$

The product rule

A product is not the product of the derivatives. Use the **product rule** 乘积法则:

$$(uv)' = u'v + uv'.$$

> “Derivative of the first times the second, plus the first times the derivative of the second.”

Differentiate $f(x) = x^2 \sin x$

$u = x^2$ ($u' = 2x$), $v = \sin x$ ($v' = \cos x$):

$$f'(x) = 2x \sin x + x^2 \cos x.$$

Read the structure first – a product needs the product rule.

The quotient rule

Quotient rule 商法则:

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}.$$

> The order matters – there is a minus sign. Write the numerator carefully.

Differentiate $\frac{x}{\cos x}$

$$\frac{1 \cdot \cos x - x \cdot (-\sin x)}{\cos^2 x}$$
$$= \frac{\cos x + x \sin x}{\cos^2 x}.$$

$\tan x$, $\cot x$, $\sec x$, $\csc x$ – derive, do not memorise

Rewrite with **identities** 恒等式, then use the quotient rule:

$$\tan x = \frac{\sin x}{\cos x}$$

$$\frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x.$$

$\tan x \to \sec^2 x$

$\cot x \to -\csc^2 x$

$\sec x \to \sec x \tan x$

$\csc x \to -\csc x \cot x$

Higher-order derivatives

Differentiating f' again gives the **second derivative** 二阶导数:

$$f''(x) = \frac{d^2y}{dx^2}.$$

It is the rate of change of the rate of change.

> “Find $k''(3)$” just means differentiate twice, then substitute.

$f(x)$
↓
$f'(x)$ slope
↓
$f''(x)$ slope of slope

Average and Instantaneous Rates of Change

> A roller coaster on a lift hill: the derivative measures instantaneous rate of change of position

4 Recap

Unit 2 – key takeaways

1 The derivative

the limit of the secant slope – the tangent's gradient

2 Existence

differentiable $\Rightarrow$ continuous; corners and vertical tangents fail

3 The rules

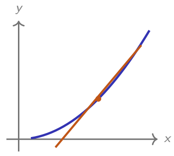
power, constant multiple, sum; product and quotient

4 Building blocks

$\sin$, $\cos$, e^x, $\ln x$ – know them by heart

Check yourself

1. Differentiate $4x^3 - 5x + 7$.
2. $f(x) = |x|$ is continuous at 0. Is it differentiable there?
3. Differentiate $x^2 \sin x$.
4. Write the tangent to $y = x^2$ at $x = 3$.



slope $= f'(a)$

Answers **1.** $12x^2 - 5$ **2.** no – a corner **3.** $2x \sin x + x^2 \cos x$ **4.** $y - 9 = 6(x - 3)$