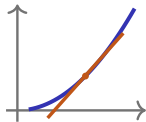


Differentiation: Definition & Properties

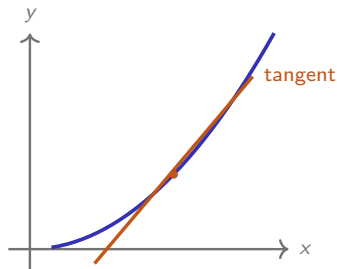
AP Calculus AB ■ Unit 2

AP Calculus AB



What we will cover

- 1 Average vs instantaneous rate
- 2 The derivative as a limit
- 3 When a derivative exists
- 4 Basic derivative rules
- 5 Product & quotient rules
- 6 Trig, exp & log derivatives



1

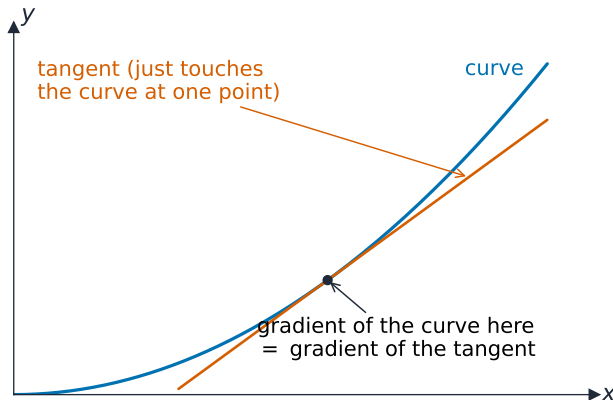
The derivative

Average vs instantaneous rate

Over an **interval**, the average rate is a **difference quotient** 差商:

$$\frac{f(a+h) - f(a)}{h} = \frac{f(x) - f(a)}{x - a}.$$

At a **point**, the **instantaneous** 瞬时 rate is the gradient of the **tangent** 切线.



From secant to tangent

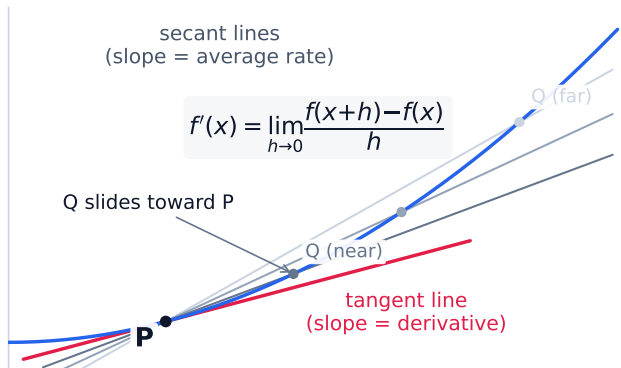
Let the point a vary and the derivative becomes a **new function**:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}.$$

This is the **definition of the derivative** (sometimes called differentiating "by first...

Its value at each x is the instantaneous rate of change there.

From secant to tangent, in detail



As Q slides toward P , each secant's slope (an average rate) approaches the tangent's slope – the derivative $f'(a)$.

The derivative as a function, and its notation

Let a vary and the derivative becomes a new function:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}.$$

This is differentiating **by first principles** 用定义求导.

$$\frac{dy}{dx} \quad f'(x) \quad y'$$

three names for the same thing

Be ready to move between **graph**, **table**, **formula** and **words**.

The tangent line – a routine exam task

The tangent at $x = a$ passes through $(a, f(a))$ with slope $f'(a)$:

$$y - f(a) = f'(a)(x - a).$$

Keep **point-slope** form ready – you need a **point** and a **slope**, nothing else.

$$f(x) = x^2 \text{ at } x = 3$$

Point: $f(3) = 9$. Slope: $f'(x) = 2x$, so $f'(3) = 6$.

$$y - 9 = 6(x - 3).$$

Exam skill – estimating f' from a table

With no formula, use a difference quotient over a **small interval around a** :

$$f'(a) \approx \frac{f(b) - f(c)}{b - c}, \quad c < a < b.$$

Full credit needs the numbers **and** the **units** 单位 – output per input.

“Approximate $M'(7.5)$ over $5 \leq t \leq 10$ ”

$$M'(7.5) \approx \frac{M(10) - M(5)}{10 - 5}$$

Values on **both sides** of 7.5 give the best estimate. Answer in **units of M per unit of t** .

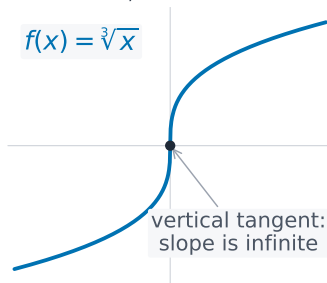
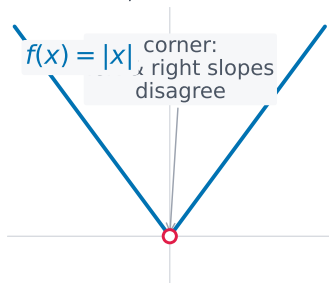
2

When a derivative exists

Differentiable $\Rightarrow$ continuous (not the reverse)

If f is **differentiable** 可导 at a point, then f is **continuous** 连续 there. A continuous function can still **fail** to be differentiable:

continuous, not differentiable continuous, not differentiable



f differentiable at $a \Rightarrow f$ continuous at a ; converse fails at corners and vertical tangents

Use the contrapositive: if f is **not continuous** at a , it is **not differentiable** at a .

The notations, and the tangent line

the notations 记号 $\frac{dy}{dx}$, $f'(x)$, y' – all naming the same object

which to use Leibniz $\frac{dy}{dx}$ when the variables matter; Lagrange $f'(x)$ when the function does

the second derivative $\frac{d^2y}{dx^2}$ or $f''(x)$

the tangent line give it as $y - f(a) = f'(a)(x - a)$

why that form it needs no rearrangement and no risk of an arithmetic slip in the intercept

the check substitute $x = a$: the equation must give $y = f(a)$

Writing the tangent in point-slope form is a marked convention, not a preference. Converting it to $y = mx + c$ adds a step and an opportunity to lose a mark.

Differentiability is the stronger condition

the relationship

differentiable $\Rightarrow$ continuous

why

a derivative is a limit that cannot exist where the function jumps

the converse is false

a continuous function can fail to be differentiable

a corner

$|x|$ at $x = 0$ – the left and right slopes differ

a cusp

$x^{2/3}$ at $x = 0$ – the slopes go to $+\infty$ and $-\infty$

a vertical tangent

$x^{1/3}$ at $x = 0$ – the slope is infinite, so no finite derivative

On a justification question, say **which direction** of the implication you are using. “It is continuous, therefore differentiable” is the error the question is set to catch.

3

Rules

The power rule

The **power rule** 幂法则 handles **any** real power:

$$\frac{d}{dx} x^r = r x^{r-1}.$$

Rewrite as a power **first**: $\frac{1}{x} = x^{-1}$,
 $\sqrt{x} = x^{1/2}$.

$$x^5 \rightarrow 5x^4$$

$$\frac{1}{x} = x^{-1} \rightarrow -x^{-2}$$

$$\sqrt{x} = x^{1/2} \rightarrow \frac{1}{2}x^{-1/2}$$

Constant, multiple, sum & difference

Constant

$$\frac{d}{dx}k = 0$$

Constant multiple

$$\frac{d}{dx}(kf) = kf'$$

Sum / difference

$$(f \pm g)' = f' \pm g'$$

Differentiate a polynomial term by term

$$\frac{d}{dx}(4x^3 - 5x + 7) = 12x^2 - 5.$$

The constant 7 contributes nothing – it does not change.

Four building blocks to know by heart

$$\frac{d}{dx} \sin x = \cos x$$

$$\frac{d}{dx} \cos x = -\sin x$$

$$\frac{d}{dx} e^x = e^x$$

$$\frac{d}{dx} \ln x = \frac{1}{x}$$

Note the **minus sign** on $\cos$, and that e^x is its **own** derivative.

A limit that is really a derivative

If a limit has the **shape** of the definition

$$\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h},$$

do not expand it – just evaluate $f'(a)$.

Evaluate the limit

$$\lim_{h \rightarrow 0} \frac{\sin\left(\frac{\pi}{2} + h\right) - 1}{h}$$

This is $\frac{d}{dx} \sin x \Big|_{x=\pi/2} = \cos \frac{\pi}{2} = 0$.

The product rule

A product is **not** the product of the derivatives. Use the **product rule** 乘积法则:

$$(uv)' = u'v + uv'.$$

“Derivative of the first times the second, **plus** the first times the derivative of the second.”

Differentiate $f(x) = x^2 \sin x$

$$u = x^2 \quad (u' = 2x), \quad v = \sin x \quad (v' = \cos x):$$

$$f'(x) = 2x \sin x + x^2 \cos x.$$

Read the **structure** first – a product needs the product rule.

The quotient rule

Quotient rule 商法则:

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}.$$

The **order matters** – there is a minus sign. Write the numerator carefully.

Differentiate $\frac{x}{\cos x}$

$$\begin{aligned} & \frac{1 \cdot \cos x - x \cdot (-\sin x)}{\cos^2 x} \\ &= \frac{\cos x + x \sin x}{\cos^2 x}. \end{aligned}$$

$\tan x$, $\cot x$, $\sec x$, $\csc x$ – derive, do not memorise

Rewrite with **identities** 恒等式, then use the quotient rule:

$$\tan x = \frac{\sin x}{\cos x}$$

$$\frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x.$$

$$\tan x \rightarrow \sec^2 x$$

$$\cot x \rightarrow -\csc^2 x$$

$$\sec x \rightarrow \sec x \tan x$$

$$\csc x \rightarrow -\csc x \cot x$$

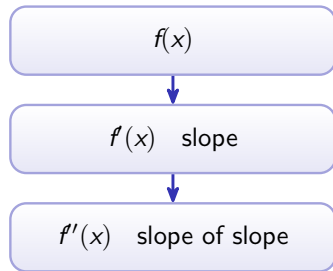
Higher-order derivatives

Differentiating f' again gives the **second derivative** 二阶导数:

$$f''(x) = \frac{d^2 y}{dx^2}.$$

It is the **rate of change of the rate of change**.

“Find $k''(3)$ ” just means **differentiate twice**, then substitute.



Average and Instantaneous Rates of Change



A roller coaster on a lift hill: the derivative measures instantaneous rate of change of position

4

Recap

Unit 2 – key takeaways

1

The derivative

the limit of the secant slope – the tangent's gradient

2

Existence

differentiable $\Rightarrow$ continuous;
corners and vertical tangents fail

3

The rules

power, constant multiple, sum;
product and quotient

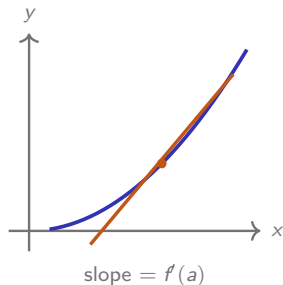
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Building blocks

$\sin$, $\cos$, e^x , $\ln x$ – know them by heart

Check yourself

- 1 Differentiate $4x^3 - 5x + 7$.
- 2 $f(x) = |x|$ is continuous at 0. Is it differentiable there?
- 3 Differentiate $x^2 \sin x$.
- 4 Write the tangent to $y = x^2$ at $x = 3$.



Answers 1. $12x^2 - 5$ 2. no – a corner 3. $2x \sin x + x^2 \cos x$ 4. $y - 9 = 6(x - 3)$