Differential Equations

AP Calculus AB

Modeling Situations with Differential Equations

A **differential equation** 微分方程 is an equation that relates a function to its own derivatives. It describes a situation by its *rate of change*. For example, "the rate of change of a quantity is proportional to its size" becomes

$$\frac{dy}{dt} = ky.$$

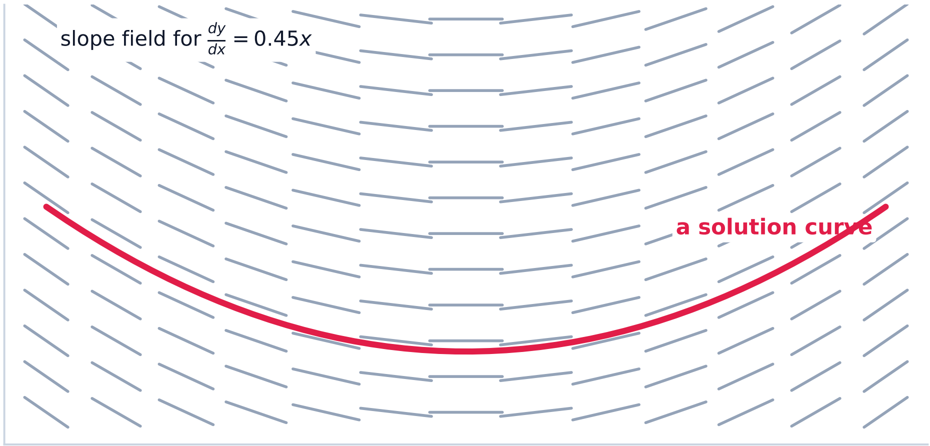
Learning to translate a sentence about a rate into a differential equation is the first skill of this unit.

Verifying Solutions for Differential Equations

A **solution** 解 of a differential equation is a function that makes it true. You can **verify** a proposed solution by differentiating it and substituting into the equation: if both sides match, it is a solution. Note a differential equation usually has **infinitely many** solutions – a whole family – differing by a constant.

Sketching Slope Fields

A **slope field** 斜率场 draws the differential equation as a grid of short segments: at each point (x, y) the segment has slope $\dfrac{dy}{dx}$ evaluated there. To sketch one, plug several points into the right-hand side and draw a small segment with that slope at each. The picture shows the shape of the solution curves without solving.



each segment is the local slope $\frac{dy}{dx}$; a solution curve is tangent to every segment it meets

Each short segment has the slope $\dfrac{dy}{dx}$ at that point; a solution curve threads through, staying tangent to the field.

Exam skill. A common part gives a portion of a slope field and asks you to sketch the particular solution through a given point – start at the point and follow the segments, staying tangent to them.

Reasoning Using Slope Fields

Solutions to a differential equation are **functions or families of functions**. Read a slope field to reason about behavior: where the segments are flat ($\frac{dy}{dx} = 0$) the solution is momentarily level; where they steepen the solution rises or falls faster; horizontal rows of equal slope suggest the rate depends only on y (or only on x).

Finding General Solutions Using Separation of Variables

Many exam differential equations are solved by **separation of variables** 分离变量法. If $\dfrac{dy}{dx}$ factors into a function of x times a function of y, move all y's to one side and all x's to the other, then integrate both sides:

$$\frac{dy}{dx} = g(x)h(y) \implies \int \frac{dy}{h(y)} = \int g(x)\,dx.$$

Add the constant of integration once (on the x-side). This gives the **general solution** 通解.

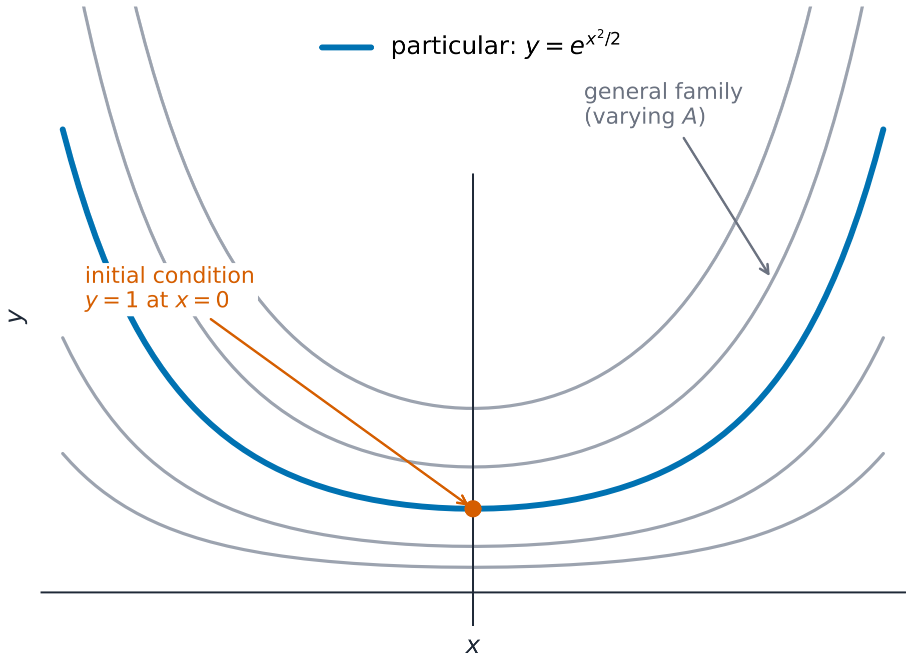
Worked example. Solve $\dfrac{dy}{dx} = xy$ with the initial condition $y(0) = 2$. Separate and integrate:

$$\int \frac{dy}{y} = \int x\,dx \;\Rightarrow\; \ln|y| = \frac{x^2}{2} + C \;\Rightarrow\; y = Ae^{x^2/2}.$$

Applying $y(0) = 2$ gives $A = 2$, so the particular solution is $y = 2e^{x^2/2}$.

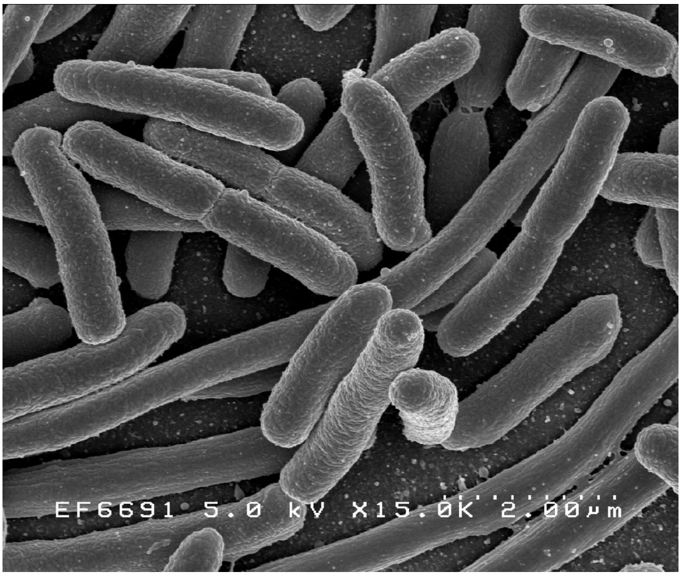
Finding Particular Solutions Using Initial Conditions and Separation of Variables

An **initial condition** 初始条件 – a known point (x_0, y_0) – pins down one curve from the family. Substitute it to solve for the constant C; the result is the **particular solution** 特解. There is exactly one solution through a given point. Watch for **domain restrictions** 定义域限制: keep the branch that contains the initial point (for example, the correct sign of a square root).



The constant gives a family of curves; an initial condition picks out one

Exponential Models with Differential Equations



E. coli under a microscope: differential equations model exponential growth of populations

The most important model is **exponential growth and decay** 指数增长与衰减. The equation $\frac{dy}{dt} = ky$ (rate proportional to size) has the solution:

$$y = y_0\, e^{kt},$$

where y_0 is the value at $t = 0$. Here $k > 0$ gives growth and $k < 0$ gives decay. You can derive this by separation of variables ($\int \frac{dy}{y} = \int k\, dt$), and it models processes like population growth, radioactive decay, and Newton's law of cooling. A follow-up part may ask for $\frac{d^2y}{dt^2}$: differentiate $\frac{dy}{dt} = ky$ again (using $\frac{dy}{dt} = ky$) to express it in terms of y.

Worked example. A sample decays by $\frac{dy}{dt} = -0.10\, y$ (in years) from $y_0 = 50$ g. The solution is $y = 50e^{-0.10t}$, so after 10 years $y = 50e^{-1} = 18.4$ g. Its **half-life** solves $25 = 50e^{-0.10t}$, i.e. $e^{-0.10t} = \frac{1}{2}$, giving $t = \frac{\ln 2}{0.10} = 6.9$ years – independent of the starting amount.

A cooling cup of coffee: Newton's law of cooling is a classic differential equation model

Exam tips

- Solve a **separable** equation by getting all y on one side and all x on the other, then integrating both sides (add $+C$ once).
- Use the initial condition to find C (a particular solution).
- Sketch or read a **slope field**: the little segments show $\frac{dy}{dx}$ at each point, and a solution curve follows them.
- Recognise **exponential** models $\frac{dy}{dt} = ky \Rightarrow y = Ce^{kt}$ (growth/decay).
- A differential equation gives the slope — you must integrate to recover the function.