

Contextual Applications of Differentiation

AP Calculus AB

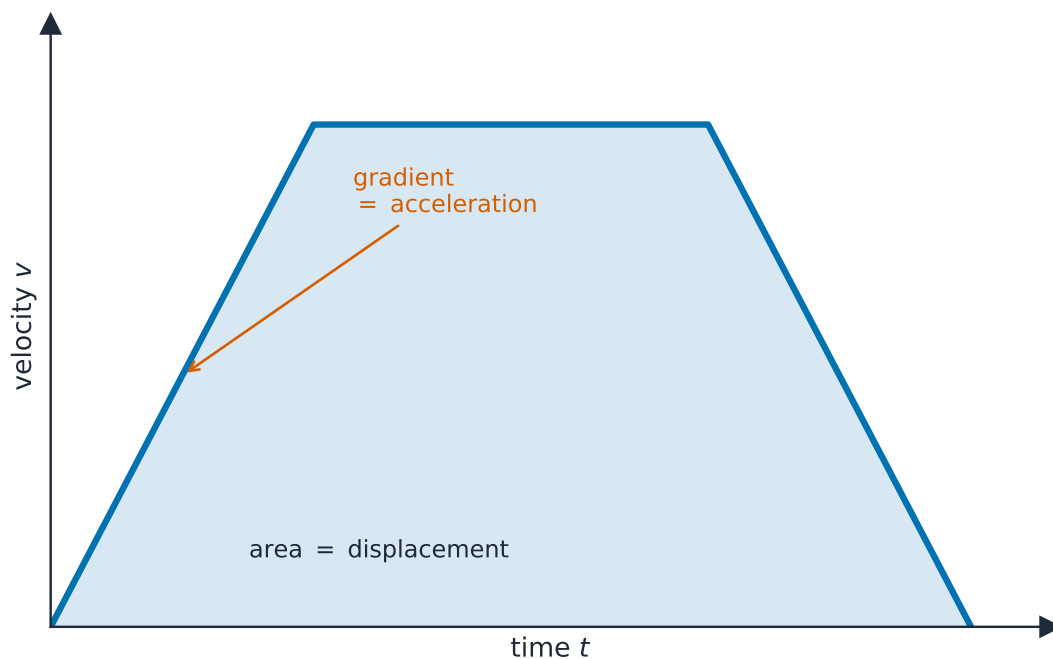
Interpreting the Derivative in Context

Once you can compute derivatives, you use them to describe the real world. The derivative $f'(x)$ is the instantaneous rate of change of f with respect to its input. Reading and reporting this rate correctly is a graded skill.

Units matter. The unit of $f'(x)$ is the unit of f divided by the unit of x . If $C(t)$ is a number of acres and t is in weeks, then $C'(t)$ is in acres per week. On the exam, "Using correct units, interpret the meaning of $g'(140)$ " wants a full sentence: the *value*, the *quantity*, the *rate word* "per", and the *moment*. For example: " $g'(140) = 2.3$ means that at $x = 140$, the quantity is increasing at about 2.3 units per unit of x ."

Straight-Line Motion: Position, Velocity, and Acceleration

For a particle moving on a line, three functions of time are linked by differentiation:



On a velocity-time graph, the area is displacement and the gradient is acceleration

- **position** 位置 $s(t)$;
- **velocity** 速度 $v(t) = s'(t)$ – signed; its sign gives direction;
- **acceleration** 加速度 $a(t) = v'(t) = s''(t)$.

Key readings (frequent exam parts):

- The particle is **at rest** 静止 when $v(t) = 0$.
- It moves **right/up** when $v(t) > 0$ and **left/down** when $v(t) < 0$; it **changes direction** where v changes sign.
- **Speed** 速率 is $|v(t)|$. Speed is *increasing* when v and a have the **same sign** (the particle is speeding up), and *decreasing* when they have **opposite signs**.

Worked example. A particle moves with $s(t) = t^3 - 6t^2 + 9t$. Then $v(t) = 3t^2 - 12t + 9 = 3(t - 1)(t - 3)$, so it is at rest at $t = 1$ and $t = 3$, and changes direction at each. At $t = 2$, $v(2) = 3(1)(-1) = -3 < 0$ (moving left) and $a(2) = 6(2) - 12 = 0$; just after, $a > 0$ while $v < 0$, so it is **slowing down** there. Distinguish carefully between velocity (has direction) and speed (does not) – the exam tests this exact difference.



A roller coaster is calculus you can feel: velocity is the derivative of position, and acceleration is the derivative of velocity

Rates of Change Beyond Motion

The same derivative idea models any changing quantity: a draining tank, a spreading population, a cooling cup. Whenever a problem says “the rate at which...”, it is describing a derivative. Read the units to know which quantity’s rate you have, then interpret in context.

Introduction to Related Rates

In a **related rates** 相关变化率 problem, several quantities change together over time, and you know some rates but want another. The engine is the chain rule: differentiate a relationship **with respect to time** t . Every variable becomes a function of t , so each derivative picks up a “ $/dt$ ” factor. Product and quotient rules may also be needed.



A rising balloon: related rates link how fast radius, volume and height change together

Solving Related Rates Problems

A reliable procedure – and a full-credit template on the exam:

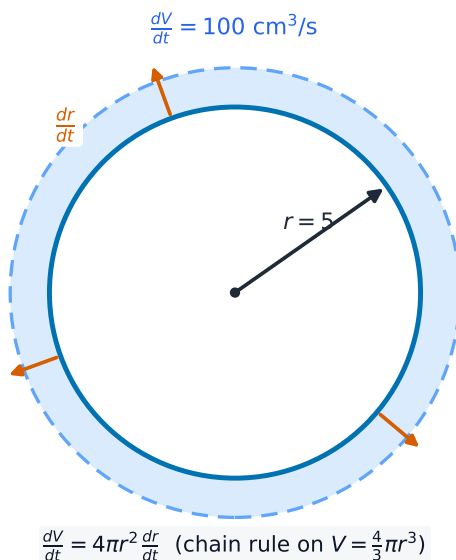
1. **Name the variables** and write down the given rates and the unknown rate (e.g. “ $\frac{dh}{dt} = -2$ cm/day, find $\frac{dV}{dt}$ ”).
2. **Write an equation** relating the quantities (often a geometric or volume formula).
3. **Differentiate both sides with respect to t** (chain rule) – *before* substituting numbers.
4. **Substitute** the known values at the instant of interest, and solve for the unknown rate.
5. State the answer with **units** and the correct **sign** (a decreasing quantity has a negative rate).

Worked example. Air is pumped into a spherical balloon so its volume grows at $\frac{dV}{dt} = 100$ cm³/s. How fast is the radius growing when $r = 5$ cm? Start from $V = \frac{4}{3}\pi r^3$

and differentiate with respect to t **first**:

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \Rightarrow 100 = 4\pi(5)^2 \frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{100}{100\pi} = \frac{1}{\pi} \approx 0.32 \text{ cm/s}.$$

Substituting $r = 5$ *too early* is the classic error – differentiate the general relationship, then plug in.



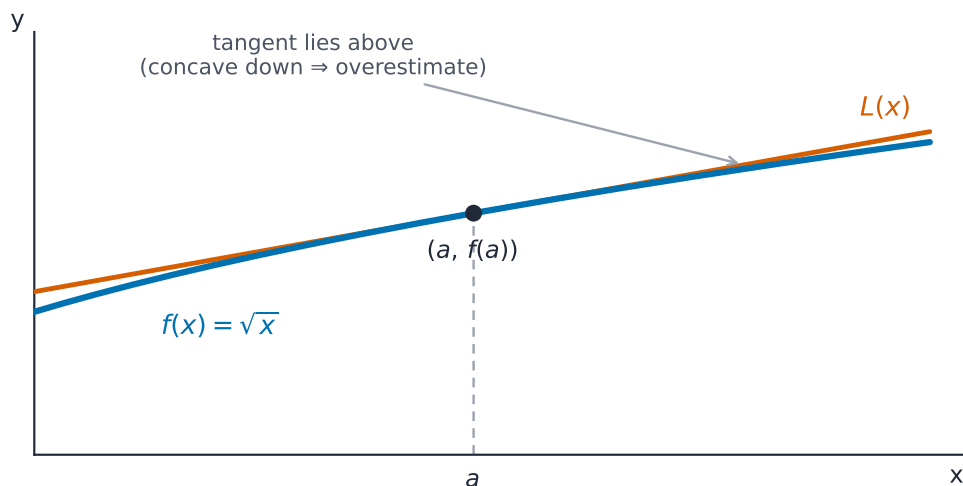
An inflating balloon links dV/dt and dr/dt through the chain rule

Local Linearity and Linearization

Near a point of tangency, a smooth curve looks like its tangent line – this is **local linearity** 局部线性. So the tangent line gives a **linear approximation** 线性近似 (linearization) of the function near that point:

$$f(x) \approx L(x) = f(a) + f'(a)(x - a).$$

Use it to estimate f at an x close to a .



$$f(x) \approx L(x) = f(a) + f'(a)(x - a) \text{ near } x = a$$

The tangent line is the best linear approximation to the curve near the point of tangency

Worked example. Estimate $\sqrt{4.1}$. Take $f(x) = \sqrt{x}$ and $a = 4$: $f(4) = 2$ and $f'(x) = \frac{1}{2\sqrt{x}}$ so $f'(4) = \frac{1}{4}$. Then $L(4.1) = 2 + \frac{1}{4}(4.1 - 4) = 2.025$ (the true value is $2.0248\dots$). Because $\sqrt{x}$ is **concave down**, the tangent lies above the curve, so this is a slight **overestimate** 高估 – justify over/under with the sign of f'' .

L'Hospital's Rule for Indeterminate Forms

When direct substitution in a quotient of limits gives the **indeterminate form** 未定式 $\frac{0}{0}$ or $\frac{\infty}{\infty}$, you may use **L'Hospital's Rule** 洛必达法则:

$$\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \lim_{x \rightarrow c} \frac{f'(x)}{g'(x)},$$

provided the right-hand limit exists. **Differentiate the top and bottom separately** (this is *not* the quotient rule), then try the limit again. First confirm the form really is $\frac{0}{0}$ or $\frac{\infty}{\infty}$ – applying the rule to any other form is a mistake.

Worked example. Evaluate $\lim_{x \rightarrow 0} \frac{\sin x}{x}$. Substituting gives $\frac{0}{0}$, so differentiate top and bottom: $\lim_{x \rightarrow 0} \frac{\cos x}{1} = \cos 0 = 1$ – confirming the famous limit from Unit 1.

Exam tips

- In motion problems: velocity is the derivative of position, acceleration the derivative of velocity; **speed increases when velocity and acceleration share a sign.**
- For **related rates**, differentiate the relating equation with respect to **time**, then substitute the given values last.
- Use the tangent line for a **linear approximation** near a known point; it is accurate only close by.
- Read the sign of a rate: positive means the quantities move together, negative means opposite.
- Always state units and interpret the answer in context.