

Week 13

AP Calculus AB

Homework bundle for this week

本周作业合集

exercises.lol

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AP Calculus AB

Name: _____ Score: _____

1. $V(t)$ is water volume in litres at time t minutes. The units of $V'(t)$ are...
 - ☐ litres
 - ☐ litres per minute
 - ☐ minutes
 - ☐ minutes per litre
2. Velocity is the derivative of...
 - ☐ acceleration
 - ☐ position
 - ☐ speed
 - ☐ time
3. $C(x)$ is cost in dollars for x items. The units of the marginal cost $C'(x)$ are...
 - ☐ dollars
 - ☐ dollars per item
 - ☐ items
 - ☐ items per dollar
4. A related-rates problem links the rates of quantities that change with respect to...
 - ☐ each other only
 - ☐ time
 - ☐ temperature
 - ☐ nothing
5. With $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$, $r = 5$, and $\frac{dr}{dt} = 2$, find $\frac{dV}{dt}$ as a multiple of π (enter the number before π).

6. The linearization of f at a is $L(x) =$
 - ☐ $f(a) + f'(a)(x - a)$
 - ☐ $f'(a) + f(a)(x - a)$
 - ☐ $f(a) \cdot f'(a)$
 - ☐ $f(x) - f(a)$

7. L'Hospital's Rule applies to **which** indeterminate forms?

☐ $\frac{0}{0}$

☐ $\frac{\infty}{\infty}$

☐ $\frac{5}{2}$

☐ $\frac{3}{0}$

Tick every correct option.

8. For $v(t) = 3t^2 - 12t + 9$, the object is at rest when $v = 0$. One such time is $t = 1$; find the **other** time.

9. For $V(t) = 100 - 4t^2$, the rate is $V'(t) = -8t$. Find $V'(3)$ (litres per minute).

10. Using $L(x) = 2 + \frac{1}{4}(x - 4)$, estimate $\sqrt{4.1} = L(4.1)$.

AP Calculus AB1. **litres per minute**

Output per input: litres per minute.

2. **position**

$v = s'$: velocity is the derivative of position.

3. **dollars per item**

Output per input: dollars per item.

4. **time**

The variables are functions of time, linked via the chain rule.

5. **200**

$$4\pi(25)(2) = 200\pi.$$

6. $f(a) + f'(a)(x - a)$

Point-slope of the tangent: $f(a) + f'(a)(x - a)$.

7. $\frac{0}{0}; \frac{\infty}{\infty}$

Only $\frac{0}{0}$ and $\frac{\infty}{\infty}$. $\frac{5}{2}$ is already determinate; $\frac{3}{0}$ is an asymptote, not indeterminate.

8. **3**

$$3(t^2 - 4t + 3) = 3(t - 1)(t - 3) = 0 \Rightarrow t = 1 \text{ or } t = 3.$$

9. **-24**

$$V'(3) = -8(3) = -24; \text{ draining at } 24 \text{ L/min.}$$

10. **2.025**

$$2 + \frac{1}{4}(0.1) = 2.025.$$

4.1 Interpreting the Derivative in Context

Name: _____ Class: _____ Date: _____

Total: 9 marks

KEY WORDS units 单位

Objective

Build the skills to answer exam questions on **interpreting the meaning of the derivative in context**.

You must be able to:

- give the **units** 单位 of a derivative
- interpret its sign and value in a real context

1 Worked examples

Study these first. Each one shows the method for a question type used later.

■ The derivative in context

A derivative is a **rate of change** with units: if $V(t)$ is a volume in cm^3 at time t (s), then $V'(t)$ is in cm^3/s .

- $f'(t) > 0$: the quantity is increasing;
- $f'(t) < 0$: it is decreasing.

2 Practice

2.1 State the units of $f'(t)$ if f is in metres and t in seconds. [1]

2.2 State what a positive derivative means. [1]

2.3 If $P(t)$ is a population and $P'(5) = 200$, interpret this. [2]

3 Exam-style questions

3.1 If $h(t)$ is a height in m and t in s, then $h'(t)$ has units [1]

- **A** m
 - **B** m/s
 - **C** s/m
 - **D** m · s
-

3.2 A positive $f'(t)$ means the quantity is [1]

- **A** decreasing
 - **B** increasing
 - **C** constant
 - **D** zero
-

3.3 $W(t)$ is water in litres at time t (min), and $W'(3) = -4$.

(a) State the units. [1]

(b) State what the sign means. [1]

(c) Interpret $W'(3) = -4$. [1]

Solutions

2.1 metres per second (m/s).

2.2 the quantity is increasing.

2.3 at $t = 5$, the population is growing at 200 per unit time.

3.1 B.

3.2 B.

3.3 (a) litres per minute (L/min). (b) the amount is decreasing. (c) at $t = 3$ min, the water is falling at 4 L/min.

t (seconds)	0	60	90	120	135	150
$f(t)$ (gallons per second)	0	0.1	0.15	0.1	0.05	0

1. A customer at a gas station is pumping gasoline into a gas tank. The rate of flow of gasoline is modeled by a differentiable function f , where $f(t)$ is measured in gallons per second and t is measured in seconds since pumping began. Selected values of $f(t)$ are given in the table.

- (a) Using correct units, interpret the meaning of $\int_{60}^{135} f(t) dt$ in the context of the problem. Use a right

Riemann sum with the three subintervals $[60, 90]$, $[90, 120]$, and $[120, 135]$ to approximate the value of

$$\int_{60}^{135} f(t) dt.$$

- (b) Must there exist a value of c , for $60 < c < 120$, such that $f'(c) = 0$? Justify your answer.

- (c) The rate of flow of gasoline, in gallons per second, can also be modeled by $g(t) = \left(\frac{t}{500}\right) \cos\left(\left(\frac{t}{120}\right)^2\right)$ for

$0 \leq t \leq 150$. Using this model, find the average rate of flow of gasoline over the time interval $0 \leq t \leq 150$.

Show the setup for your calculations.

- (d) Using the model g defined in part (c), find the value of $g'(140)$. Interpret the meaning of your answer in the context of the problem.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

r (centimeters)	0	1	2	2.5	4
$f(r)$ (milligrams per square centimeter)	1	2	6	10	18

1. The density of a bacteria population in a circular petri dish at a distance r centimeters from the center of the dish is given by an increasing, differentiable function f , where $f(r)$ is measured in milligrams per square centimeter. Values of $f(r)$ for selected values of r are given in the table above.
- (a) Use the data in the table to estimate $f'(2.25)$. Using correct units, interpret the meaning of your answer in the context of this problem.
- (b) The total mass, in milligrams, of bacteria in the petri dish is given by the integral expression $2\pi \int_0^4 r f(r) dr$. Approximate the value of $2\pi \int_0^4 r f(r) dr$ using a right Riemann sum with the four subintervals indicated by the data in the table.
- (c) Is the approximation found in part (b) an overestimate or underestimate of the total mass of bacteria in the petri dish? Explain your reasoning.
- (d) The density of bacteria in the petri dish, for $1 \leq r \leq 4$, is modeled by the function g defined by $g(r) = 2 - 16(\cos(1.57\sqrt{r}))^3$. For what value of k , $1 < k < 4$, is $g(k)$ equal to the average value of $g(r)$ on the interval $1 \leq r \leq 4$?

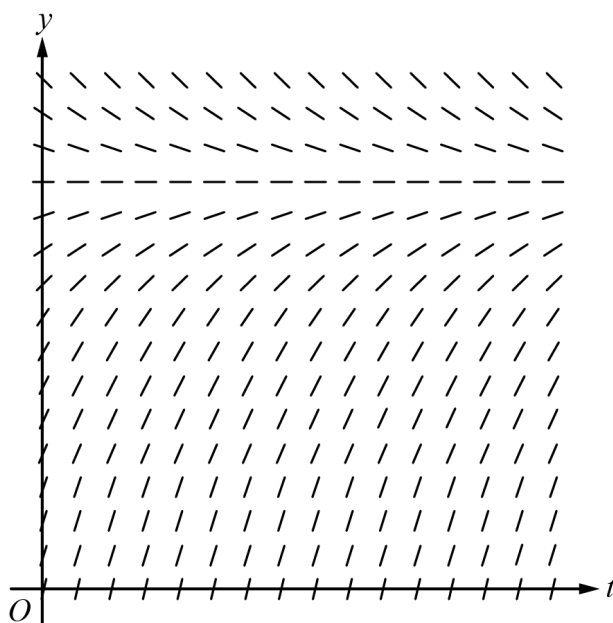
Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

A medication is administered to a patient. The amount, in milligrams, of the medication in the patient at

time t hours is modeled by a function $y = A(t)$ that satisfies the differential equation $\frac{dy}{dt} = \frac{12 - y}{3}$. At time

$t = 0$ hours, there are 0 milligrams of the medication in the patient.

- (a) A portion of the slope field for the differential equation $\frac{dy}{dt} = \frac{12 - y}{3}$ is given below. Sketch the solution curve through the point $(0, 0)$.



- (b) Using correct units, interpret the statement $\lim_{t \rightarrow \infty} A(t) = 12$ in the context of this problem.

- (c) Use separation of variables to find $y = A(t)$, the particular solution to the differential equation

$$\frac{dy}{dt} = \frac{12 - y}{3} \text{ with initial condition } A(0) = 0.$$

- (d) A different procedure is used to administer the medication to a second patient. The amount, in milligrams, of the medication in the second patient at time t hours is modeled by a function $y = B(t)$ that satisfies the differential equation $\frac{dy}{dt} = 3 - \frac{y}{t+2}$. At time $t = 1$ hour, there are 2.5 milligrams of the medication in the second patient. Is the rate of change of the amount of medication in the second patient increasing or decreasing at time $t = 1$? Give a reason for your answer.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

STOP

END OF EXAM

t (years)	2	3	5	7	10
$H(t)$ (meters)	1.5	2	6	11	15

4. The height of a tree at time t is given by a twice-differentiable function H , where $H(t)$ is measured in meters and t is measured in years. Selected values of $H(t)$ are given in the table above.
- (a) Use the data in the table to estimate $H'(6)$. Using correct units, interpret the meaning of $H'(6)$ in the context of the problem.
- (b) Explain why there must be at least one time t , for $2 < t < 10$, such that $H'(t) = 2$.
- (c) Use a trapezoidal sum with the four subintervals indicated by the data in the table to approximate the average height of the tree over the time interval $2 \leq t \leq 10$.
- (d) The height of the tree, in meters, can also be modeled by the function G , given by $G(x) = \frac{100x}{1+x}$, where x is the diameter of the base of the tree, in meters. When the tree is 50 meters tall, the diameter of the base of the tree is increasing at a rate of 0.03 meter per year. According to this model, what is the rate of change of the height of the tree with respect to time, in meters per year, at the time when the tree is 50 meters tall?
-

2. When a certain grocery store opens, it has 50 pounds of bananas on a display table. Customers remove bananas from the display table at a rate modeled by

$$f(t) = 10 + (0.8t)\sin\left(\frac{t^3}{100}\right) \text{ for } 0 < t \leq 12,$$

where $f(t)$ is measured in pounds per hour and t is the number of hours after the store opened. After the store has been open for three hours, store employees add bananas to the display table at a rate modeled by

$$g(t) = 3 + 2.4\ln(t^2 + 2t) \text{ for } 3 < t \leq 12,$$

where $g(t)$ is measured in pounds per hour and t is the number of hours after the store opened.

- (a) How many pounds of bananas are removed from the display table during the first 2 hours the store is open?
- (b) Find $f'(7)$. Using correct units, explain the meaning of $f'(7)$ in the context of the problem.
- (c) Is the number of pounds of bananas on the display table increasing or decreasing at time $t = 5$? Give a reason for your answer.
- (d) How many pounds of bananas are on the display table at time $t = 8$?
-

END OF PART A OF SECTION II

CALCULUS AB
SECTION II, Part A**Time—30 minutes****Number of problems—2****A graphing calculator is required for these problems.**

1. Grass clippings are placed in a bin, where they decompose. For $0 \leq t \leq 30$, the amount of grass clippings remaining in the bin is modeled by $A(t) = 6.687(0.931)^t$, where $A(t)$ is measured in pounds and t is measured in days.
- (a) Find the average rate of change of $A(t)$ over the interval $0 \leq t \leq 30$. Indicate units of measure.
- (b) Find the value of $A'(15)$. Using correct units, interpret the meaning of the value in the context of the problem.
- (c) Find the time t for which the amount of grass clippings in the bin is equal to the average amount of grass clippings in the bin over the interval $0 \leq t \leq 30$.
- (d) For $t > 30$, $L(t)$, the linear approximation to A at $t = 30$, is a better model for the amount of grass clippings remaining in the bin. Use $L(t)$ to predict the time at which there will be 0.5 pound of grass clippings remaining in the bin. Show the work that leads to your answer.
-

4.2 Straight-Line Motion: Position, Velocity, and Acceleration

Name: _____ Class: _____ Date: _____

Total: 9 marks

KEY WORDS velocity 速度 • position 位置 • acceleration 加速度 • speed 速率 • at rest 静止

Objective

Build the skills to answer exam questions on **straight-line motion: position, velocity, and acceleration**.

You must be able to:

- relate **position** 位置, **velocity** 速度, and **acceleration** 加速度 by differentiation
- find when an object is at rest

1 Worked examples

Study these first. Each one shows the method for a question type used later.

■ Straight-line motion

- velocity $v(t) = x'(t)$;
- acceleration $a(t) = v'(t) = x''(t)$;
- **speed** $= |v(t)|$.

The object is **at rest** when $v = 0$; it moves in the positive direction when $v > 0$.

■ Example

$$x(t) = t^2 - 4t: v(t) = 2t - 4 = 0 \text{ at } t = 2.$$

2 Practice

2.1 State velocity in terms of position. [1]

2.2 State acceleration in terms of velocity. [1]

2.3 For $x(t) = t^2 - 4t$, find $v(t)$ and the time at rest. [2]

3 Exam-style questions

3.1 Velocity is the derivative of [1]

- **A** acceleration
 - **B** position
 - **C** speed
 - **D** distance
-

3.2 The object is at rest when [1]

- **A** $a = 0$
 - **B** $v = 0$
 - **C** $x = 0$
 - **D** $t = 0$
-

3.3 $x(t) = t^2 - 4t$.

(a) Find $v(t)$. [1]

(b) Find $a(t)$. [1]

(c) Find when $v = 0$.

[1]

Solutions

2.1 $v(t) = x'(t)$.

2.2 $a(t) = v'(t)$.

2.3 $v(t) = 2t - 4$; at rest when $2t - 4 = 0$, i.e. $t = 2$.

3.1 B.

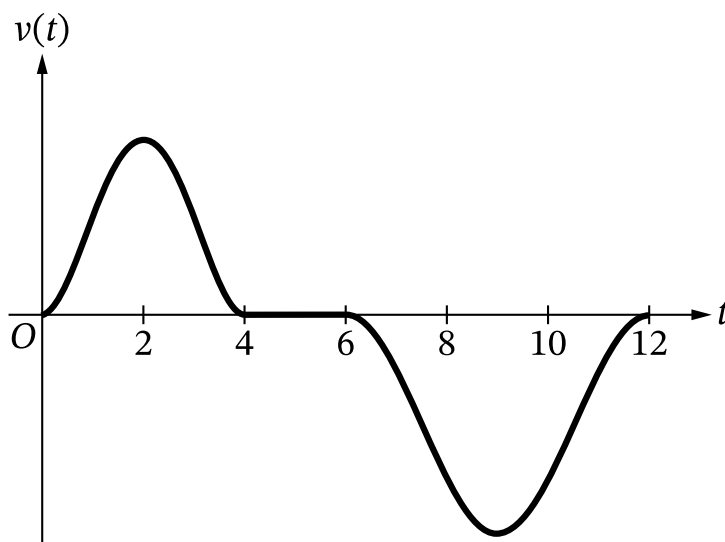
3.2 B.

3.3 (a) $v(t) = 2t - 4$. (b) $a(t) = 2$. (c) $t = 2$.

5. A remote-controlled toy car moves back and forth along a straight path so that its velocity at time t is given by the function v , where $v(t)$ is measured in feet per second and t is measured in seconds.

$$v(t) = \begin{cases} t^4 - 8t^3 + 16t^2 & \text{for } 0 \leq t \leq 4 \\ 0 & \text{for } 4 < t < 6 \\ 10 \cos\left(\frac{\pi}{3}t\right) - 10 & \text{for } 6 \leq t \leq 12 \end{cases}$$

The graph of $v(t)$ is shown.



Part A

Find the acceleration of the car at time $t = 1$ second. Show the work that leads to your answer.

Part B

Is the car speeding up or slowing down at time $t = 1$ second? Give a reason for your answer.

Part C

Find the distance, in feet, that the car traveled over the time interval $0 \leq t \leq 4$ seconds. Show the work that leads to your answer.

Part D

Find the average velocity of the car over the time interval $6 \leq t \leq 12$ seconds. Show the work that leads to your answer.

5. Two particles, H and J , are moving along the x -axis. For $0 \leq t \leq 5$, the position of particle H at time t is given by $x_H(t) = e^{t^2 - 4t}$ and the velocity of particle J at time t is given by $v_J(t) = 2t(t^2 - 1)^3$.
- A. Find the velocity of particle H at time $t = 1$. Show the work that leads to your answer.
- B. During what open intervals of time t , for $0 < t < 5$, are particles H and J moving in opposite directions? Give a reason for your answer.
- C. It can be shown that $v_J'(2) > 0$. Is the speed of particle J increasing, decreasing, or neither at time $t = 2$? Give a reason for your answer.
- D. Particle J is at position $x = 7$ at time $t = 0$. Find the position of particle J at time $t = 2$. Show the work that leads to your answer.

2. A particle moves along the x -axis so that its velocity at time $t \geq 0$ is given by $v(t) = \ln(t^2 - 4t + 5) - 0.2t$.
- (a) There is one time, $t = t_R$, in the interval $0 < t < 2$ when the particle is at rest (not moving). Find t_R . For $0 < t < t_R$, is the particle moving to the right or to the left? Give a reason for your answer.
- (b) Find the acceleration of the particle at time $t = 1.5$. Show the setup for your calculations. Is the speed of the particle increasing or decreasing at time $t = 1.5$? Explain your reasoning.
- (c) The position of the particle at time t is $x(t)$, and its position at time $t = 1$ is $x(1) = -3$. Find the position of the particle at time $t = 4$. Show the setup for your calculations.
- (d) Find the total distance traveled by the particle over the interval $1 \leq t \leq 4$. Show the setup for your calculations.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

Name:

AP Calculus AB: **2024**

END OF PART A

CALCULUS AB

SECTION II, Part B

Time—1 hour

4 Questions

NO CALCULATOR IS ALLOWED FOR THESE QUESTIONS.

2. Stephen swims back and forth along a straight path in a 50-meter-long pool for 90 seconds. Stephen's velocity is modeled by $v(t) = 2.38e^{-0.02t}\sin\left(\frac{\pi}{56}t\right)$, where t is measured in seconds and $v(t)$ is measured in meters per second.
- (a) Find all times t in the interval $0 < t < 90$ at which Stephen changes direction. Give a reason for your answer.
- (b) Find Stephen's acceleration at time $t = 60$ seconds. Show the setup for your calculations, and indicate units of measure. Is Stephen speeding up or slowing down at time $t = 60$ seconds? Give a reason for your answer.
- (c) Find the distance between Stephen's position at time $t = 20$ seconds and his position at time $t = 80$ seconds. Show the setup for your calculations.
- (d) Find the total distance Stephen swims over the time interval $0 \leq t \leq 90$ seconds. Show the setup for your calculations.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

END OF PART A

CALCULUS AB**SECTION II, Part B****Time—1 hour****4 Questions****NO CALCULATOR IS ALLOWED FOR THESE QUESTIONS.**

AP Calculus AB: 2022

6. Particle P moves along the x -axis such that, for time $t > 0$, its position is given by $x_P(t) = 6 - 4e^{-t}$.

Particle Q moves along the y -axis such that, for time $t > 0$, its velocity is given by $v_Q(t) = \frac{1}{t^2}$. At time $t = 1$, the position of particle Q is $y_Q(1) = 2$.

- (a) Find $v_P(t)$, the velocity of particle P at time t .
- (b) Find $a_Q(t)$, the acceleration of particle Q at time t . Find all times t , for $t > 0$, when the speed of particle Q is decreasing. Justify your answer.
- (c) Find $y_Q(t)$, the position of particle Q at time t .
- (d) As $t \rightarrow \infty$, which particle will eventually be farther from the origin? Give a reason for your answer.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

STOP**END OF EXAM**

AP Calculus AB: 2021

A particle, P , is moving along the x -axis. The velocity of particle P at time t is given by $v_P(t) = \sin(t^{1.5})$ for $0 \leq t \leq \pi$. At time $t = 0$, particle P is at position $x = 5$.

A second particle, Q , also moves along the x -axis. The velocity of particle Q at time t is given by $v_Q(t) = (t - 1.8) \cdot 1.25^t$ for $0 \leq t \leq \pi$. At time $t = 0$, particle Q is at position $x = 10$.

- (a) Find the positions of particles P and Q at time $t = 1$.
- (b) Are particles P and Q moving toward each other or away from each other at time $t = 1$? Explain your reasoning.
- (c) Find the acceleration of particle Q at time $t = 1$. Is the speed of particle Q increasing or decreasing at time $t = 1$? Explain your reasoning.
- (d) Find the total distance traveled by particle P over the time interval $0 \leq t \leq \pi$.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

Name:

AP Calculus AB: **2021**

END OF PART A

CALCULUS AB
SECTION II, Part B

Time—1 hour

4 Questions

NO CALCULATOR IS ALLOWED FOR THESE QUESTIONS.

2. A particle moves along the x -axis with velocity given by $v(t) = \frac{10 \sin(0.4t^2)}{t^2 - t + 3}$ for time $0 \leq t \leq 3.5$.

The particle is at position $x = -5$ at time $t = 0$.

(a) Find the acceleration of the particle at time $t = 3$.

(b) Find the position of the particle at time $t = 3$.

(c) Evaluate $\int_0^{3.5} v(t) \, dt$, and evaluate $\int_0^{3.5} |v(t)| \, dt$. Interpret the meaning of each integral in the context of the problem.

(d) A second particle moves along the x -axis with position given by $x_2(t) = t^2 - t$ for $0 \leq t \leq 3.5$. At what time t are the two particles moving with the same velocity?

END OF PART A OF SECTION II

5. Two particles move along the x -axis. For $0 \leq t \leq 8$, the position of particle P at time t is given by

$$x_P(t) = \ln(t^2 - 2t + 10), \text{ while the velocity of particle } Q \text{ at time } t \text{ is given by } v_Q(t) = t^2 - 8t + 15.$$

Particle Q is at position $x = 5$ at time $t = 0$.

- (a) For $0 \leq t \leq 8$, when is particle P moving to the left?
 - (b) For $0 \leq t \leq 8$, find all times t during which the two particles travel in the same direction.
 - (c) Find the acceleration of particle Q at time $t = 2$. Is the speed of particle Q increasing, decreasing, or neither at time $t = 2$? Explain your reasoning.
 - (d) Find the position of particle Q the first time it changes direction.
-

2. For $t \geq 0$, a particle moves along the x -axis. The velocity of the particle at time t is given by

$$v(t) = 1 + 2 \sin\left(\frac{t^2}{2}\right). \text{ The particle is at position } x = 2 \text{ at time } t = 4.$$

- (a) At time $t = 4$, is the particle speeding up or slowing down?
 - (b) Find all times t in the interval $0 < t < 3$ when the particle changes direction. Justify your answer.
 - (c) Find the position of the particle at time $t = 0$.
 - (d) Find the total distance the particle travels from time $t = 0$ to time $t = 3$.
-

END OF PART A OF SECTION II

CALCULUS AB
SECTION II, Part B**Time—60 minutes****Number of problems—4****No calculator is allowed for these problems.**

t (minutes)	0	12	20	24	40
$v(t)$ (meters per minute)	0	200	240	−220	150

3. Johanna jogs along a straight path. For $0 \leq t \leq 40$, Johanna's velocity is given by a differentiable function v . Selected values of $v(t)$, where t is measured in minutes and $v(t)$ is measured in meters per minute, are given in the table above.

(a) Use the data in the table to estimate the value of $v'(16)$.

(b) Using correct units, explain the meaning of the definite integral $\int_0^{40} |v(t)| dt$ in the context of the problem.

Approximate the value of $\int_0^{40} |v(t)| dt$ using a right Riemann sum with the four subintervals indicated in the table.

(c) Bob is riding his bicycle along the same path. For $0 \leq t \leq 10$, Bob's velocity is modeled by $B(t) = t^3 - 6t^2 + 300$, where t is measured in minutes and $B(t)$ is measured in meters per minute.

Find Bob's acceleration at time $t = 5$.

(d) Based on the model B from part (c), find Bob's average velocity during the interval $0 \leq t \leq 10$.

4.3 Rates of Change Beyond Motion

Name: _____ Class: _____ Date: _____

Total: 9 marks

KEY WORDS marginal cost 边际成本

Objective

Build the skills to answer exam questions on **rates of change in applied contexts other than motion**.

You must be able to:

- interpret a derivative as a rate in any context (e.g. **marginal cost** 边际成本)

1 Worked examples

Study these first. Each one shows the method for a question type used later.

■ Rates in other contexts

A derivative measures the rate of change of **any** quantity: marginal cost $C'(x)$, a flow rate, a reaction rate. Interpret its sign and units just as for motion.

■ Example

If $C(x)$ is cost in dollars for x items, $C'(100) = 5$ means about \$5 to produce the 101st item.

2 Practice

2.1 State what marginal cost is, as a derivative. [1]

2.2 State the units of $C'(x)$ if C is in dollars and x in items. [1]

2.3 If $C'(100) = 5$, interpret this. [2]

3 Exam-style questions

3.1 Marginal cost is the derivative of [1]

- **A** revenue
 - **B** the cost function
 - **C** profit
 - **D** price
-

3.2 If $A'(t) > 0$, the area A is [1]

- **A** shrinking
 - **B** growing
 - **C** constant
 - **D** zero
-

3.3 $C(x)$ is cost in dollars for x items, and $C'(50) = 8$.

(a) State the units. [1]

(b) State what the sign means. [1]

(c) Interpret $C'(50) = 8$. [1]

Solutions

2.1 the derivative of the cost function, $C'(x)$.

2.2 dollars per item.

2.3 near $x = 100$ items, cost rises about \$5 per additional item.

3.1 B.

3.2 B.

3.3 (a) dollars per item. (b) cost is rising with output. (c) about \$8 to produce the 51st item.

4.4 Introduction to Related Rates

Name: _____ Class: _____ Date: _____

Total: 9 marks

KEY WORDS time 时间 • related rates 相关变化率

Objective

Build the skills to answer exam questions on **introduction to related rates**.

You must be able to:

- differentiate a relation with respect to **time** 时间 to link two rates

1 Worked examples

Study these first. Each one shows the method for a question type used later.

■ Related rates

When two quantities are related, their rates are related: differentiate the relation with respect to t (using the chain rule).

■ Example

$$A = \pi r^2 \Rightarrow \frac{dA}{dt} = 2\pi r \frac{dr}{dt}.$$

2 Practice

2.1 State the first step in a related-rates problem. [1]

2.2 Differentiate $A = \pi r^2$ with respect to t . [2]

2.3 State what links the two rates. [1]

3 Exam-style questions

3.1 In related rates, you differentiate the relation with respect to [1]

- **A** x
 - **B** time t
 - **C** r
 - **D** nothing
-

3.2 Differentiating $A = \pi r^2$ with respect to t gives [1]

- **A** $2\pi r$
 - **B** $2\pi r \frac{dr}{dt}$
 - **C** πr^2
 - **D** $2r$
-

3.3 $V = s^3$ (a cube).

(a) Differentiate with respect to t . [1]

(b) State the chain-rule factor. [1]

(c) Write $\frac{dV}{dt}$. [1]

Solutions

2.1 write the relation connecting the quantities.

2.2 $\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$.

2.3 the derivative of the relation with respect to t .

3.1 B.

3.2 B.

3.3 (a) $\frac{dV}{dt} = 3s^2 \frac{ds}{dt}$. (b) $\frac{ds}{dt}$. (c) $\frac{dV}{dt} = 3s^2 \frac{ds}{dt}$.

5. Consider the curve defined by the equation $x^2 + 3y + 2y^2 = 48$. It can be shown that $\frac{dy}{dx} = \frac{-2x}{3 + 4y}$.
- (a) There is a point on the curve near $(2, 4)$ with x -coordinate 3. Use the line tangent to the curve at $(2, 4)$ to approximate the y -coordinate of this point.
- (b) Is the horizontal line $y = 1$ tangent to the curve? Give a reason for your answer.
- (c) The curve intersects the positive x -axis at the point $(\sqrt{48}, 0)$. Is the line tangent to the curve at this point vertical? Give a reason for your answer.
- (d) For time $t \geq 0$, a particle is moving along another curve defined by the equation $y^3 + 2xy = 24$. At the instant the particle is at the point $(4, 2)$, the y -coordinate of the particle's position is decreasing at a rate of 2 units per second. At that instant, what is the rate of change of the x -coordinate of the particle's position with respect to time?

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

t (years)	2	3	5	7	10
$H(t)$ (meters)	1.5	2	6	11	15

4. The height of a tree at time t is given by a twice-differentiable function H , where $H(t)$ is measured in meters and t is measured in years. Selected values of $H(t)$ are given in the table above.
- (a) Use the data in the table to estimate $H'(6)$. Using correct units, interpret the meaning of $H'(6)$ in the context of the problem.
- (b) Explain why there must be at least one time t , for $2 < t < 10$, such that $H'(t) = 2$.
- (c) Use a trapezoidal sum with the four subintervals indicated by the data in the table to approximate the average height of the tree over the time interval $2 \leq t \leq 10$.
- (d) The height of the tree, in meters, can also be modeled by the function G , given by $G(x) = \frac{100x}{1+x}$, where x is the diameter of the base of the tree, in meters. When the tree is 50 meters tall, the diameter of the base of the tree is increasing at a rate of 0.03 meter per year. According to this model, what is the rate of change of the height of the tree with respect to time, in meters per year, at the time when the tree is 50 meters tall?
-

4.5 Solving Related Rates Problems

Name: _____ Class: _____ Date: _____

Total: 9 marks

KEY WORDS related rates 相关变化率

Objective

Build the skills to answer exam questions on **solving related rates problems**.**You must be able to:**

- carry out the full related-rates procedure and solve for the unknown rate

1 Worked examples

Study these first. Each one shows the method for a question type used later.

■ Solving related rates

1. write the relation;
2. differentiate with respect to t ;
3. substitute the known values and rates;
4. solve for the unknown rate.

■ Example

A circle's radius grows at $\frac{dr}{dt} = 2$ cm/s. At $r = 3$: $\frac{dA}{dt} = 2\pi r \frac{dr}{dt} = 2\pi(3)(2) = 12\pi$ cm²/s.

2 Practice

2.1 List the steps of a related-rates solution.

[1]

2.2 A circle's r grows at 2 cm/s. At $r = 3$, find $\frac{dA}{dt}$ (with $A = \pi r^2$).

[2]

2.3 State the units of $\frac{dA}{dt}$ here. [1]

3 Exam-style questions

3.1 After differentiating, the next step is to [1]

- **A** stop
 - **B** substitute known values and solve
 - **C** integrate
 - **D** graph
-

3.2 If $\frac{dr}{dt} = 2$ and $r = 3$, then $\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$ equals [1]

- **A** 6π
 - **B** 12π
 - **C** 4π
 - **D** 9π
-

3.3 $A = \pi r^2$, $\frac{dr}{dt} = 2$, $r = 3$.

(a) Write $\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$. [1]

(b) Substitute. [1]

(c) State $\frac{dA}{dt}$. [1]

Solutions

2.1 write the relation; differentiate with respect to t ; substitute; solve for the unknown rate.

2.2 $\frac{dA}{dt} = 2\pi(3)(2) = 12\pi \text{ cm}^2/\text{s}.$

2.3 $\text{cm}^2/\text{s}.$

3.1 B.

3.2 B.

3.3 (a) $\frac{dA}{dt} = 2\pi r \frac{dr}{dt}.$ (b) $2\pi(3)(2).$ (c) $12\pi \text{ cm}^2/\text{s}.$

6. Consider the curve G defined by the equation $y^3 - y^2 - y + \frac{1}{4}x^2 = 0$.

- A. Show that $\frac{dy}{dx} = \frac{-x}{2(3y^2 - 2y - 1)}$.
- B. There is a point P on the curve G near $(2, -1)$ with x -coordinate 1.6. Use the line tangent to the curve at $(2, -1)$ to approximate the y -coordinate of point P .
- C. For $x > 0$ and $y > 0$, there is a point S on the curve G at which the line tangent to the curve at that point is vertical. Find the y -coordinate of point S . Show the work that leads to your answer.
- D. A particle moves along the curve H defined by the equation $2xy + \ln y = 8$. At the instant when the particle is at the point $(4, 1)$, $\frac{dx}{dt} = 3$. Find $\frac{dy}{dt}$ at that instant. Show the work that leads to your answer.

STOP

END OF EXAM

5. Consider the curve defined by the equation $x^2 + 3y + 2y^2 = 48$. It can be shown that $\frac{dy}{dx} = \frac{-2x}{3 + 4y}$.

- (a) There is a point on the curve near $(2, 4)$ with x -coordinate 3. Use the line tangent to the curve at $(2, 4)$ to approximate the y -coordinate of this point.
- (b) Is the horizontal line $y = 1$ tangent to the curve? Give a reason for your answer.
- (c) The curve intersects the positive x -axis at the point $(\sqrt{48}, 0)$. Is the line tangent to the curve at this point vertical? Give a reason for your answer.
- (d) For time $t \geq 0$, a particle is moving along another curve defined by the equation $y^3 + 2xy = 24$. At the instant the particle is at the point $(4, 2)$, the y -coordinate of the particle's position is decreasing at a rate of 2 units per second. At that instant, what is the rate of change of the x -coordinate of the particle's position with respect to time?

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

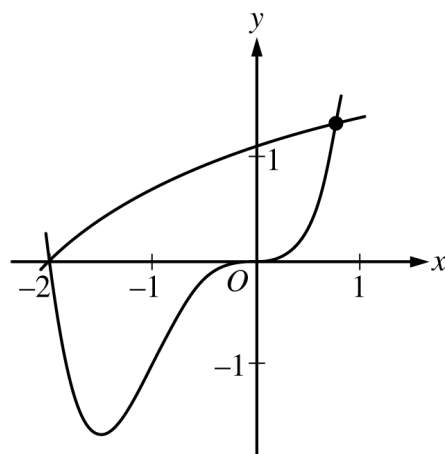
6. Consider the curve given by the equation $6xy = 2 + y^5$.

- (a) Show that $\frac{dy}{dx} = \frac{2y}{y^2 - 2x}$.
- (b) Find the coordinates of a point on the curve at which the line tangent to the curve is horizontal, or explain why no such point exists.
- (c) Find the coordinates of a point on the curve at which the line tangent to the curve is vertical, or explain why no such point exists.
- (d) A particle is moving along the curve. At the instant when the particle is at the point $\left(\frac{1}{2}, -2\right)$, its horizontal position is increasing at a rate of $\frac{dx}{dt} = \frac{2}{3}$ unit per second. What is the value of $\frac{dy}{dt}$, the rate of change of the particle's vertical position, at that instant?

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

STOP

END OF EXAM



2. Let f and g be the functions defined by $f(x) = \ln(x+3)$ and $g(x) = x^4 + 2x^3$. The graphs of f and g , shown in the figure above, intersect at $x = -2$ and $x = B$, where $B > 0$.
- (a) Find the area of the region enclosed by the graphs of f and g .
- (b) For $-2 \leq x \leq B$, let $h(x)$ be the vertical distance between the graphs of f and g . Is h increasing or decreasing at $x = -0.5$? Give a reason for your answer.
- (c) The region enclosed by the graphs of f and g is the base of a solid. Cross sections of the solid taken perpendicular to the x -axis are squares. Find the volume of the solid.
- (d) A vertical line in the xy -plane travels from left to right along the base of the solid described in part (c). The vertical line is moving at a constant rate of 7 units per second. Find the rate of change of the area of the cross section above the vertical line with respect to time when the vertical line is at position $x = -0.5$.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

Name:

AP Calculus AB: **2022**

END OF PART A

CALCULUS AB
SECTION II, Part B

Time—1 hour

4 Questions

NO CALCULATOR IS ALLOWED FOR THESE QUESTIONS.

t (days)	0	3	7	10	12
$r'(t)$ (centimeters per day)	-6.1	-5.0	-4.4	-3.8	-3.5

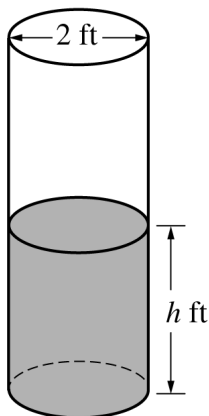
4. An ice sculpture melts in such a way that it can be modeled as a cone that maintains a conical shape as it decreases in size. The radius of the base of the cone is given by a twice-differentiable function r , where $r(t)$ is measured in centimeters and t is measured in days. The table above gives selected values of $r'(t)$, the rate of change of the radius, over the time interval $0 \leq t \leq 12$.

- (a) Approximate $r''(8.5)$ using the average rate of change of r' over the interval $7 \leq t \leq 10$. Show the computations that lead to your answer, and indicate units of measure.
- (b) Is there a time t , $0 \leq t \leq 3$, for which $r'(t) = -6$? Justify your answer.
- (c) Use a right Riemann sum with the four subintervals indicated in the table to approximate the value of

$$\int_0^{12} r'(t) dt.$$

- (d) The height of the cone decreases at a rate of 2 centimeters per day. At time $t = 3$ days, the radius is 100 centimeters and the height is 50 centimeters. Find the rate of change of the volume of the cone with respect to time, in cubic centimeters per day, at time $t = 3$ days. (The volume V of a cone with radius r and height h is $V = \frac{1}{3}\pi r^2 h$.)

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.



4. A cylindrical barrel with a diameter of 2 feet contains collected rainwater, as shown in the figure above. The water drains out through a valve (not shown) at the bottom of the barrel. The rate of change of the height h of the water in the barrel with respect to time t is modeled by $\frac{dh}{dt} = -\frac{1}{10}\sqrt{h}$, where h is measured in feet and t is measured in seconds. (The volume V of a cylinder with radius r and height h is $V = \pi r^2 h$.)
- (a) Find the rate of change of the volume of water in the barrel with respect to time when the height of the water is 4 feet. Indicate units of measure.
- (b) When the height of the water is 3 feet, is the rate of change of the height of the water with respect to time increasing or decreasing? Explain your reasoning.
- (c) At time $t = 0$ seconds, the height of the water is 5 feet. Use separation of variables to find an expression for h in terms of t .
-

t (years)	2	3	5	7	10
$H(t)$ (meters)	1.5	2	6	11	15

4. The height of a tree at time t is given by a twice-differentiable function H , where $H(t)$ is measured in meters and t is measured in years. Selected values of $H(t)$ are given in the table above.
- (a) Use the data in the table to estimate $H'(6)$. Using correct units, interpret the meaning of $H'(6)$ in the context of the problem.
- (b) Explain why there must be at least one time t , for $2 < t < 10$, such that $H'(t) = 2$.
- (c) Use a trapezoidal sum with the four subintervals indicated by the data in the table to approximate the average height of the tree over the time interval $2 \leq t \leq 10$.
- (d) The height of the tree, in meters, can also be modeled by the function G , given by $G(x) = \frac{100x}{1+x}$, where x is the diameter of the base of the tree, in meters. When the tree is 50 meters tall, the diameter of the base of the tree is increasing at a rate of 0.03 meter per year. According to this model, what is the rate of change of the height of the tree with respect to time, in meters per year, at the time when the tree is 50 meters tall?
-

CALCULUS AB
SECTION II, Part A

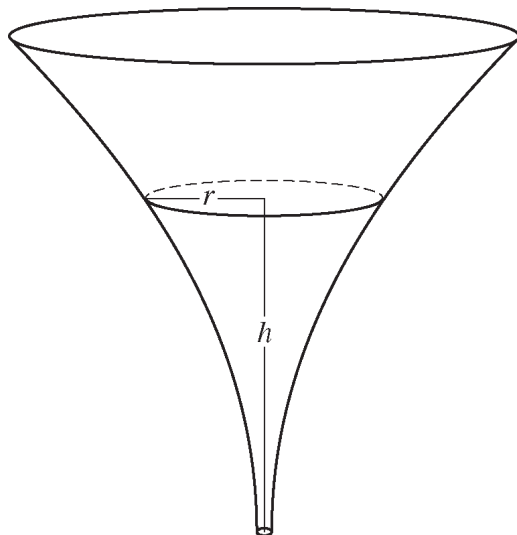
Time—30 minutes

Number of questions—2

A GRAPHING CALCULATOR IS REQUIRED FOR THESE QUESTIONS.

h (feet)	0	2	5	10
$A(h)$ (square feet)	50.3	14.4	6.5	2.9

1. A tank has a height of 10 feet. The area of the horizontal cross section of the tank at height h feet is given by the function A , where $A(h)$ is measured in square feet. The function A is continuous and decreases as h increases. Selected values for $A(h)$ are given in the table above.
- (a) Use a left Riemann sum with the three subintervals indicated by the data in the table to approximate the volume of the tank. Indicate units of measure.
- (b) Does the approximation in part (a) overestimate or underestimate the volume of the tank? Explain your reasoning.
- (c) The area, in square feet, of the horizontal cross section at height h feet is modeled by the function f given by $f(h) = \frac{50.3}{e^{0.2h} + h}$. Based on this model, find the volume of the tank. Indicate units of measure.
- (d) Water is pumped into the tank. When the height of the water is 5 feet, the height is increasing at the rate of 0.26 foot per minute. Using the model from part (c), find the rate at which the volume of water is changing with respect to time when the height of the water is 5 feet. Indicate units of measure.
-



5. The inside of a funnel of height 10 inches has circular cross sections, as shown in the figure above. At height h , the radius of the funnel is given by $r = \frac{1}{20}(3 + h^2)$, where $0 \leq h \leq 10$. The units of r and h are inches.
- (a) Find the average value of the radius of the funnel.
 - (b) Find the volume of the funnel.
 - (c) The funnel contains liquid that is draining from the bottom. At the instant when the height of the liquid is $h = 3$ inches, the radius of the surface of the liquid is decreasing at a rate of $\frac{1}{5}$ inch per second. At this instant, what is the rate of change of the height of the liquid with respect to time?
-

t (minutes)	0	2	5	8	12
$v_A(t)$ (meters/minute)	0	100	40	-120	-150

4. Train A runs back and forth on an east-west section of railroad track. Train A 's velocity, measured in meters per minute, is given by a differentiable function $v_A(t)$, where time t is measured in minutes. Selected values for $v_A(t)$ are given in the table above.
- (a) Find the average acceleration of train A over the interval $2 \leq t \leq 8$.
- (b) Do the data in the table support the conclusion that train A 's velocity is -100 meters per minute at some time t with $5 < t < 8$? Give a reason for your answer.
- (c) At time $t = 2$, train A 's position is 300 meters east of the Origin Station, and the train is moving to the east. Write an expression involving an integral that gives the position of train A , in meters from the Origin Station, at time $t = 12$. Use a trapezoidal sum with three subintervals indicated by the table to approximate the position of the train at time $t = 12$.
- (d) A second train, train B , travels north from the Origin Station. At time t the velocity of train B is given by $v_B(t) = -5t^2 + 60t + 25$, and at time $t = 2$ the train is 400 meters north of the station. Find the rate, in meters per minute, at which the distance between train A and train B is changing at time $t = 2$.
-

4.6 Local Linearity and Linearization

Name: _____ Class: _____ Date: _____

Total: 9 marks

KEY WORDS linearization 线性化 • local linearity 局部线性

Objective

Build the skills to answer exam questions on **local linearity and linearization**.**You must be able to:**

- write the **linearization** 线性化 $L(x) = f(a) + f'(a)(x - a)$
- use it to approximate a value

1 Worked examples

Study these first. Each one shows the method for a question type used later.

■ Linearization

Near $x = a$,

$$f(x) \approx L(x) = f(a) + f'(a)(x - a)$$

— the tangent line at a .

■ Example

 $f(x) = \sqrt{x}$ at $a = 4$: $f(4) = 2$, $f'(4) = \frac{1}{4}$, so $L(x) = 2 + \frac{1}{4}(x - 4)$ and $\sqrt{4.1} \approx 2.025$.

2 Practice

2.1 Write the linearization formula. [1]

2.2 For $f(x) = \sqrt{x}$ at $a = 4$, state $f(4)$ and $f'(4)$. [2]

2.3 State what $L(x)$ approximates. [1]

3 Exam-style questions

3.1 The linearization of f at a is [1]

- **A** $f(a)$
 - **B** $f(a) + f'(a)(x - a)$
 - **C** $f'(a)(x - a)$
 - **D** $f(x) \cdot a$
-

3.2 Linearization uses the _____ line. [1]

- **A** secant
 - **B** tangent
 - **C** normal
 - **D** horizontal
-

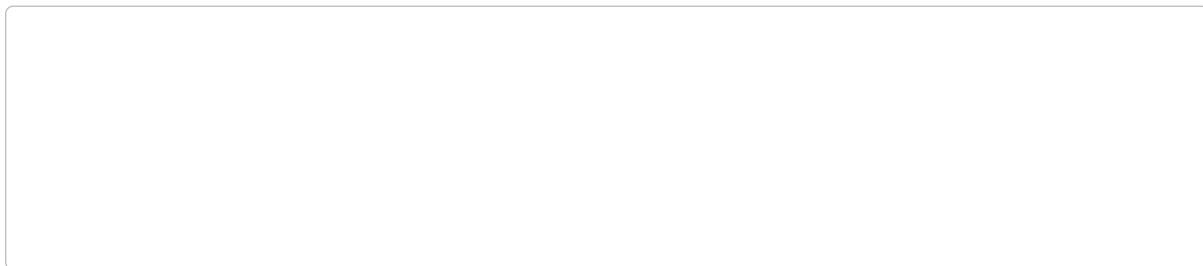
3.3 $f(x) = x^2$, $a = 3$, so $f(3) = 9$, $f'(3) = 6$.

(a) Write $L(x)$. [1]

(b) Use it for $x = 3.1$. [1]

(c) State $L(3.1)$.

[1]



Solutions

2.1 $L(x) = f(a) + f'(a)(x - a).$

2.2 $f(4) = 2; f'(4) = \frac{1}{4}.$

2.3 the value of $f(x)$ for x near a .

3.1 B.

3.2 B.

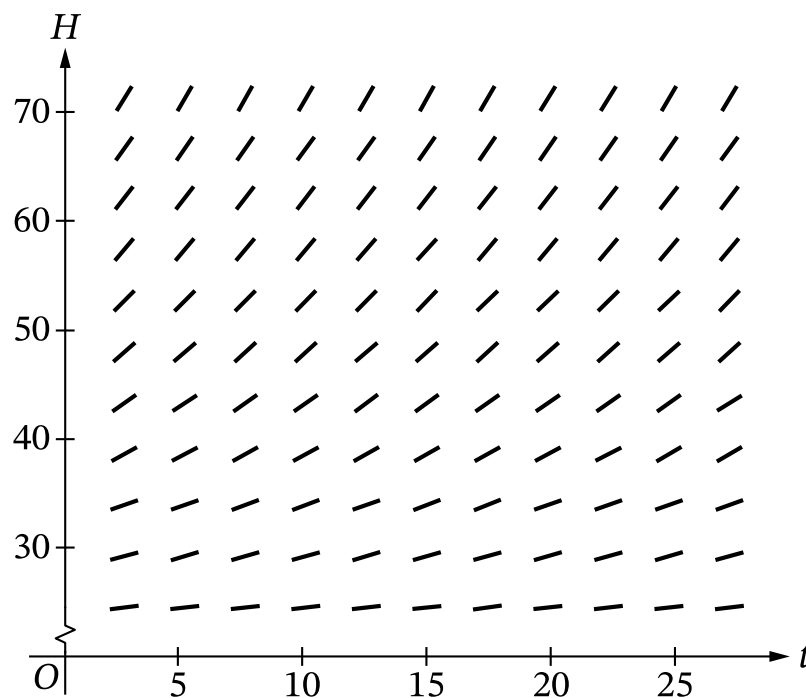
3.3 (a) $L(x) = 9 + 6(x - 3).$ (b) $9 + 6(0.1).$ (c) 9.6.

3. A pie is taken from a hot oven and put on a table. The internal temperature of the pie at time t minutes can be modeled by the function H that satisfies the differential equation $\frac{dH}{dt} = -\frac{1}{15}(H - 20)$, where $H(t)$ is measured in degrees Celsius and $H(0) = 75$. For $t > 0$, it is known that $20 < H(t) < 75$.

Part A

Explain why the following could not be a slope field for the differential equation

$$\frac{dH}{dt} = -\frac{1}{15}(H - 20).$$

**Part B**

Find the slope of the line tangent to the graph of H at time $t = 0$. Show the work that leads to your answer.

Part C

It can be shown that $\frac{d^2H}{dt^2} = \frac{1}{225}(H - 20)$. The line tangent to the graph of H at time $t = 0$ is used to approximate $H(5)$, the internal temperature of the pie at time $t = 5$. Is this approximation an overestimate or an underestimate for the actual value of $H(5)$? Give a reason for your answer.

Part D

Use separation of variables to find an expression for $H(t)$, the particular solution to the given differential equation with initial condition $H(0) = 75$.

6. Consider the curve G defined by the equation $y^3 - y^2 - y + \frac{1}{4}x^2 = 0$.

- A. Show that $\frac{dy}{dx} = \frac{-x}{2(3y^2 - 2y - 1)}$.
- B. There is a point P on the curve G near $(2, -1)$ with x -coordinate 1.6. Use the line tangent to the curve at $(2, -1)$ to approximate the y -coordinate of point P .
- C. For $x > 0$ and $y > 0$, there is a point S on the curve G at which the line tangent to the curve at that point is vertical. Find the y -coordinate of point S . Show the work that leads to your answer.
- D. A particle moves along the curve H defined by the equation $2xy + \ln y = 8$. At the instant when the particle is at the point $(4, 1)$, $\frac{dx}{dt} = 3$. Find $\frac{dy}{dt}$ at that instant. Show the work that leads to your answer.

STOP

END OF EXAM

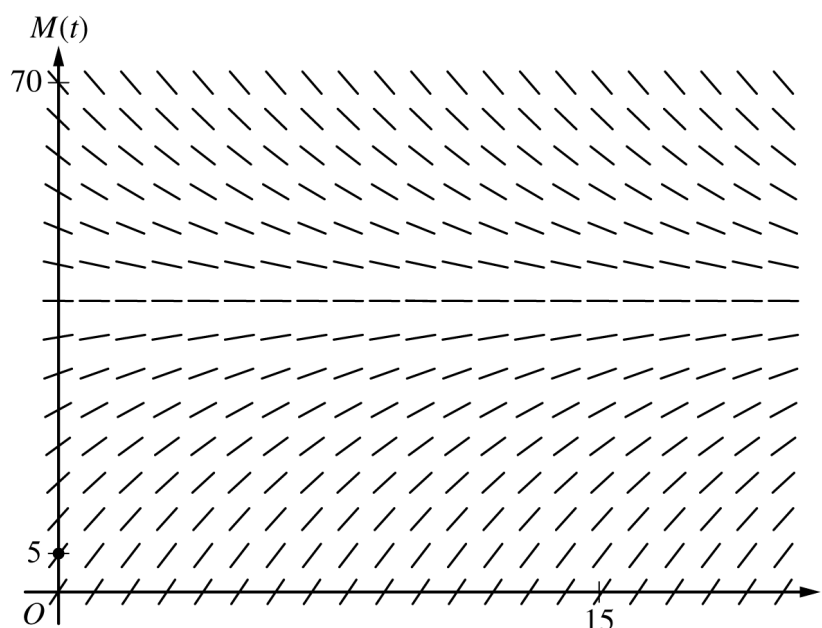
5. Consider the curve defined by the equation $x^2 + 3y + 2y^2 = 48$. It can be shown that $\frac{dy}{dx} = \frac{-2x}{3 + 4y}$.

- (a) There is a point on the curve near $(2, 4)$ with x -coordinate 3. Use the line tangent to the curve at $(2, 4)$ to approximate the y -coordinate of this point.
- (b) Is the horizontal line $y = 1$ tangent to the curve? Give a reason for your answer.
- (c) The curve intersects the positive x -axis at the point $(\sqrt{48}, 0)$. Is the line tangent to the curve at this point vertical? Give a reason for your answer.
- (d) For time $t \geq 0$, a particle is moving along another curve defined by the equation $y^3 + 2xy = 24$. At the instant the particle is at the point $(4, 2)$, the y -coordinate of the particle's position is decreasing at a rate of 2 units per second. At that instant, what is the rate of change of the x -coordinate of the particle's position with respect to time?

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

3. A bottle of milk is taken out of a refrigerator and placed in a pan of hot water to be warmed. The increasing function M models the temperature of the milk at time t , where $M(t)$ is measured in degrees Celsius ($^{\circ}\text{C}$) and t is the number of minutes since the bottle was placed in the pan. M satisfies the differential equation $\frac{dM}{dt} = \frac{1}{4}(40 - M)$. At time $t = 0$, the temperature of the milk is 5°C . It can be shown that $M(t) < 40$ for all values of t .

- (a) A slope field for the differential equation $\frac{dM}{dt} = \frac{1}{4}(40 - M)$ is shown. Sketch the solution curve through the point $(0, 5)$.

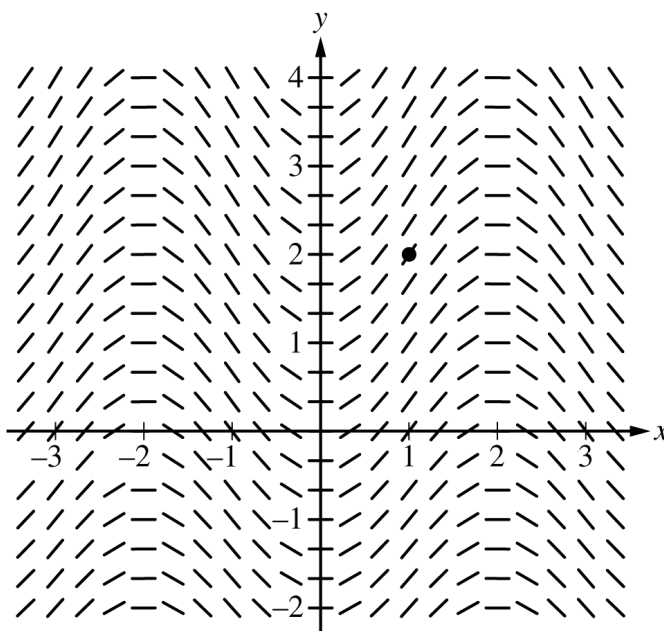


- (b) Use the line tangent to the graph of M at $t = 0$ to approximate $M(2)$, the temperature of the milk at time $t = 2$ minutes.
- (c) Write an expression for $\frac{d^2M}{dt^2}$ in terms of M . Use $\frac{d^2M}{dt^2}$ to determine whether the approximation from part (b) is an underestimate or an overestimate for the actual value of $M(2)$. Give a reason for your answer.
- (d) Use separation of variables to find an expression for $M(t)$, the particular solution to the differential equation $\frac{dM}{dt} = \frac{1}{4}(40 - M)$ with initial condition $M(0) = 5$.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

5. Consider the differential equation $\frac{dy}{dx} = \frac{1}{2} \sin\left(\frac{\pi}{2}x\right)\sqrt{y+7}$. Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(1) = 2$. The function f is defined for all real numbers.

- (a) A portion of the slope field for the differential equation is given below. Sketch the solution curve through the point $(1, 2)$.



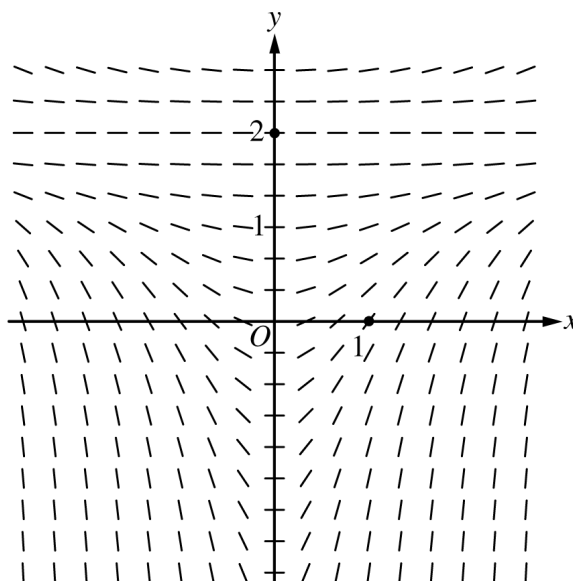
- (b) Write an equation for the line tangent to the solution curve in part (a) at the point $(1, 2)$. Use the equation to approximate $f(0.8)$.
- (c) It is known that $f''(x) > 0$ for $-1 \leq x \leq 1$. Is the approximation found in part (b) an overestimate or an underestimate for $f(0.8)$? Give a reason for your answer.
- (d) Use separation of variables to find $y = f(x)$, the particular solution to the differential equation

$$\frac{dy}{dx} = \frac{1}{2} \sin\left(\frac{\pi}{2}x\right)\sqrt{y+7} \text{ with the initial condition } f(1) = 2.$$

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

6. Consider the differential equation $\frac{dy}{dx} = \frac{1}{3}x(y - 2)^2$.

- (a) A slope field for the given differential equation is shown below. Sketch the solution curve that passes through the point $(0, 2)$, and sketch the solution curve that passes through the point $(1, 0)$.



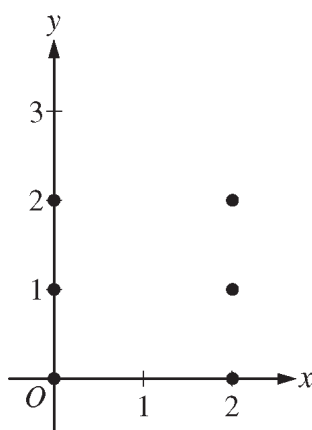
- (b) Let $y = f(x)$ be the particular solution to the given differential equation with initial condition $f(1) = 0$. Write an equation for the line tangent to the graph of $y = f(x)$ at $x = 1$. Use your equation to approximate $f(0.7)$.
- (c) Find the particular solution $y = f(x)$ to the given differential equation with initial condition $f(1) = 0$.

STOP
END OF EXAM

4. At time $t = 0$, a boiled potato is taken from a pot on a stove and left to cool in a kitchen. The internal temperature of the potato is 91°C at time $t = 0$, and the internal temperature of the potato is greater than 27°C for all times $t > 0$. The internal temperature of the potato at time t minutes can be modeled by the function H that satisfies the differential equation $\frac{dH}{dt} = -\frac{1}{4}(H - 27)$, where $H(t)$ is measured in degrees Celsius and $H(0) = 91$.
- (a) Write an equation for the line tangent to the graph of H at $t = 0$. Use this equation to approximate the internal temperature of the potato at time $t = 3$.
- (b) Use $\frac{d^2H}{dt^2}$ to determine whether your answer in part (a) is an underestimate or an overestimate of the internal temperature of the potato at time $t = 3$.
- (c) For $t < 10$, an alternate model for the internal temperature of the potato at time t minutes is the function G that satisfies the differential equation $\frac{dG}{dt} = -(G - 27)^{2/3}$, where $G(t)$ is measured in degrees Celsius and $G(0) = 91$. Find an expression for $G(t)$. Based on this model, what is the internal temperature of the potato at time $t = 3$?

4. Consider the differential equation $\frac{dy}{dx} = \frac{y^2}{x - 1}$.

- (a) On the axes provided, sketch a slope field for the given differential equation at the six points indicated.

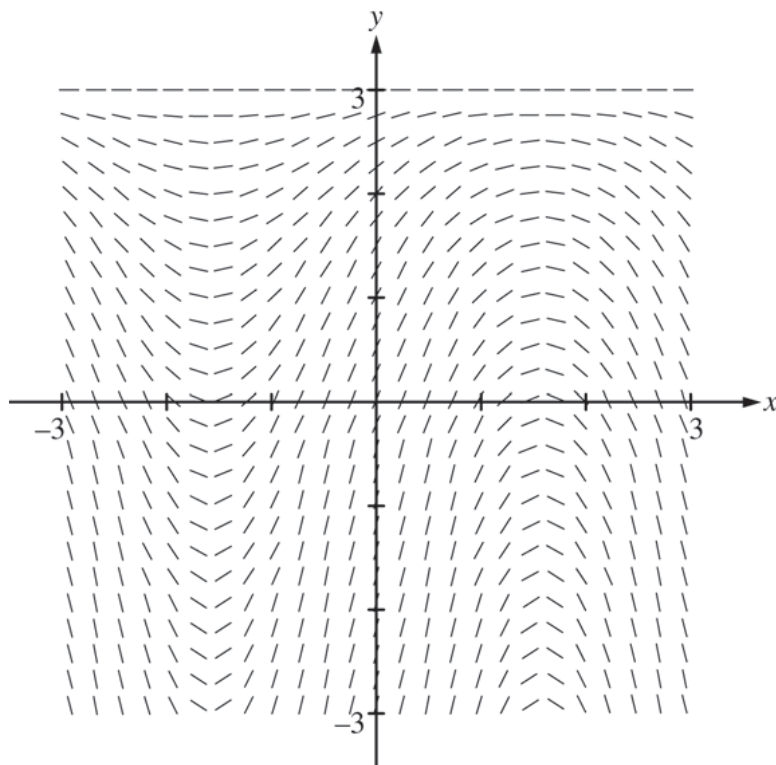


- (b) Let $y = f(x)$ be the particular solution to the given differential equation with the initial condition $f(2) = 3$. Write an equation for the line tangent to the graph of $y = f(x)$ at $x = 2$. Use your equation to approximate $f(2.1)$.
- (c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(2) = 3$.

CALCULUS AB
SECTION II, Part A**Time—30 minutes****Number of problems—2****A graphing calculator is required for these problems.**

1. Grass clippings are placed in a bin, where they decompose. For $0 \leq t \leq 30$, the amount of grass clippings remaining in the bin is modeled by $A(t) = 6.687(0.931)^t$, where $A(t)$ is measured in pounds and t is measured in days.
- (a) Find the average rate of change of $A(t)$ over the interval $0 \leq t \leq 30$. Indicate units of measure.
- (b) Find the value of $A'(15)$. Using correct units, interpret the meaning of the value in the context of the problem.
- (c) Find the time t for which the amount of grass clippings in the bin is equal to the average amount of grass clippings in the bin over the interval $0 \leq t \leq 30$.
- (d) For $t > 30$, $L(t)$, the linear approximation to A at $t = 30$, is a better model for the amount of grass clippings remaining in the bin. Use $L(t)$ to predict the time at which there will be 0.5 pound of grass clippings remaining in the bin. Show the work that leads to your answer.
-

6. Consider the differential equation $\frac{dy}{dx} = (3 - y)\cos x$. Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(0) = 1$. The function f is defined for all real numbers.
- (a) A portion of the slope field of the differential equation is given below. Sketch the solution curve through the point $(0, 1)$.



- (b) Write an equation for the line tangent to the solution curve in part (a) at the point $(0, 1)$. Use the equation to approximate $f(0.2)$.
- (c) Find $y = f(x)$, the particular solution to the differential equation with the initial condition $f(0) = 1$.

STOP

END OF EXAM

4.7 L'Hospital's Rule for Indeterminate Forms

Name: _____ Class: _____ Date: _____

Total: 9 marks

KEY WORDS l'hospital's rule 洛必达法则 • indeterminate form 未定式

Objective

Build the skills to answer exam questions on **L'Hospital's Rule for indeterminate forms**.

You must be able to:

- recognise an **indeterminate form** 未定式 $\frac{0}{0}$ or $\frac{\infty}{\infty}$
- apply **L'Hospital's Rule** 洛必达法则

1 Worked examples

Study these first. Each one shows the method for a question type used later.

■ L'Hospital's Rule

For a limit of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$,

$$\lim \frac{f}{g} = \lim \frac{f'}{g'}$$

(when the right-hand limit exists).

■ Example

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = \lim_{x \rightarrow 0} \frac{\cos x}{1} = 1.$$

2 Practice

2.1 State the forms L'Hospital's Rule applies to. [1]

2.2 Apply it to $\lim_{x \rightarrow 0} \frac{\sin x}{x}$. [2]

2.3 State what you do to the numerator and denominator. [1]

3 Exam-style questions

3.1 L'Hospital's Rule applies to [1]

- A any limit
 - B $\frac{0}{0}$ or $\frac{\infty}{\infty}$
 - C $\frac{1}{0}$
 - D $\frac{5}{2}$
-

3.2 L'Hospital's Rule replaces $\frac{f}{g}$ with [1]

- A $f \cdot g$
 - B $\frac{f'}{g'}$
 - C $\frac{f'}{g}$
 - D $\frac{f}{g'}$
-

3.3 $\lim_{x \rightarrow 0} \frac{e^x - 1}{x}$.

(a) Check the form. [1]

(b) Differentiate top and bottom. [1]

(c) Evaluate.

[1]

Solutions

2.1 $\frac{0}{0}$ and $\frac{\infty}{\infty}$.

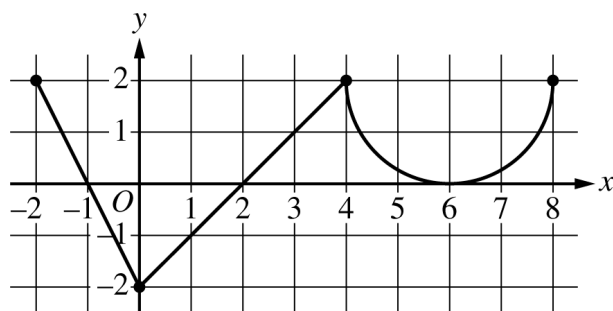
2.2 $\lim_{x \rightarrow 0} \frac{\cos x}{1} = 1$.

2.3 differentiate each separately, then take the limit of the new quotient.

3.1 B.

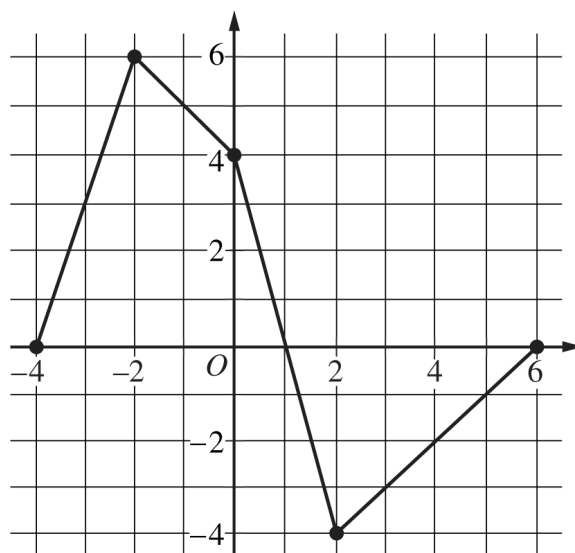
3.2 B.

3.3 (a) $\frac{0}{0}$. (b) $\frac{e^x}{1}$. (c) 1.

Graph of f'

4. The function f is defined on the closed interval $[-2, 8]$ and satisfies $f(2) = 1$. The graph of f' , the derivative of f , consists of two line segments and a semicircle, as shown in the figure.
- (a) Does f have a relative minimum, a relative maximum, or neither at $x = 6$? Give a reason for your answer.
- (b) On what open intervals, if any, is the graph of f concave down? Give a reason for your answer.
- (c) Find the value of $\lim_{x \rightarrow 2} \frac{6f(x) - 3x}{x^2 - 5x + 6}$, or show that it does not exist. Justify your answer.
- (d) Find the absolute minimum value of f on the closed interval $[-2, 8]$. Justify your answer.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

Graph of f

Let f be a continuous function defined on the closed interval $-4 \leq x \leq 6$. The graph of f , consisting of four line segments, is shown above. Let G be the function defined by $G(x) = \int_0^x f(t) \, dt$.

(a) On what open intervals is the graph of G concave up? Give a reason for your answer.

(b) Let P be the function defined by $P(x) = G(x) \cdot f(x)$. Find $P'(3)$.

(c) Find $\lim_{x \rightarrow 2} \frac{G(x)}{x^2 - 2x}$.

(d) Find the average rate of change of G on the interval $[-4, 2]$. Does the Mean Value Theorem guarantee a value c , $-4 < c < 2$, for which $G'(c)$ is equal to this average rate of change? Justify your answer.

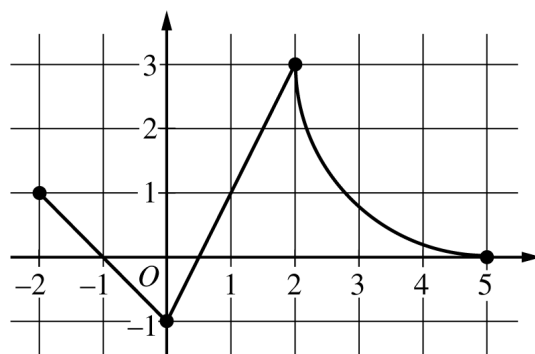
Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

CALCULUS AB
SECTION II, Part B

Time—1 hour

Number of questions—4

NO CALCULATOR IS ALLOWED FOR THESE QUESTIONS.



Graph of f

3. The continuous function f is defined on the closed interval $-6 \leq x \leq 5$. The figure above shows a portion of the graph of f , consisting of two line segments and a quarter of a circle centered at the point $(5, 3)$. It is known that the point $(3, 3 - \sqrt{5})$ is on the graph of f .
- (a) If $\int_{-6}^5 f(x) \, dx = 7$, find the value of $\int_{-6}^{-2} f(x) \, dx$. Show the work that leads to your answer.
- (b) Evaluate $\int_3^5 (2f'(x) + 4) \, dx$.
- (c) The function g is given by $g(x) = \int_{-2}^x f(t) \, dt$. Find the absolute maximum value of g on the interval $-2 \leq x \leq 5$. Justify your answer.
- (d) Find $\lim_{x \rightarrow 1} \frac{10^x - 3f'(x)}{f(x) - \arctan x}$.

6. Functions f , g , and h are twice-differentiable functions with $g(2) = h(2) = 4$. The line $y = 4 + \frac{\pi}{3}(x - 2)$ is tangent to both the graph of g at $x = 2$ and the graph of h at $x = 2$.
- (a) Find $h'(2)$.
- (b) Let a be the function given by $a(x) = 3x^3h(x)$. Write an expression for $a'(x)$. Find $a'(2)$.
- (c) The function h satisfies $h(x) = \frac{x^2 - 4}{1 - (f(x))^3}$ for $x \neq 2$. It is known that $\lim_{x \rightarrow 2} h(x)$ can be evaluated using L'Hospital's Rule. Use $\lim_{x \rightarrow 2} h(x)$ to find $f(2)$ and $f'(2)$. Show the work that leads to your answers.
- (d) It is known that $g(x) \leq h(x)$ for $1 < x < 3$. Let k be a function satisfying $g(x) \leq k(x) \leq h(x)$ for $1 < x < 3$. Is k continuous at $x = 2$? Justify your answer.
-

STOP
END OF EXAM

5. Let f be the function defined by $f(x) = e^x \cos x$.

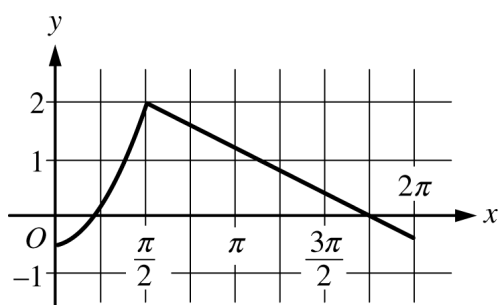
(a) Find the average rate of change of f on the interval $0 \leq x \leq \pi$.

(b) What is the slope of the line tangent to the graph of f at $x = \frac{3\pi}{2}$?

(c) Find the absolute minimum value of f on the interval $0 \leq x \leq 2\pi$. Justify your answer.

(d) Let g be a differentiable function such that $g\left(\frac{\pi}{2}\right) = 0$. The graph of g' , the derivative of g , is shown

below. Find the value of $\lim_{x \rightarrow \pi/2} \frac{f(x)}{g(x)}$ or state that it does not exist. Justify your answer.



Graph of g'