

# 8.11 Volumes by the Washer Method

Name: \_\_\_\_\_ Class: \_\_\_\_\_ Date: \_\_\_\_\_

Total: 14 marks

KEY WORDS washer 垫圈 • washer method 垫圈法 • disc 圆盘

## Objective

Build the skills to answer exam questions on the **washer method** — volumes of revolution with a hole.

**You must be able to:**

- identify the **outer** and **inner** radii of a washer (ring)
- integrate  $\pi(R_{\text{outer}}^2 - R_{\text{inner}}^2)$  (the **washer method** 垫圈法)
- set each radius as a distance from a curve to the axis

## 1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

### ■ When there is a hole

If the region does **not** touch the axis, revolving it leaves a hole, so each slice is a **washer** (a ring). Its area is (outer disc) – (inner disc):

$$V = \pi \int_a^b (R_{\text{outer}}^2 - R_{\text{inner}}^2) dx.$$

### ■ Identifying the radii

Revolve the region between  $y = \sqrt{x}$  (top) and  $y = x$  (bottom) on  $[0, 1]$  about the  $x$ -axis. Outer radius (further curve)  $R = \sqrt{x}$ ; inner radius  $r = x$ :

$$V = \pi \int_0^1 ((\sqrt{x})^2 - x^2) dx = \pi \int_0^1 (x - x^2) dx = \pi \left( \frac{1}{2} - \frac{1}{3} \right) = \frac{\pi}{6}.$$

### ■ Do not square then subtract wrongly

$R^2 - r^2$  is **not**  $(R - r)^2$ . Square each radius separately, then subtract.

### ■ Choosing outer vs inner

The **outer** radius comes from the curve **farther** from the axis; the **inner** from the curve **nearer** the axis. A quick sketch settles which is which.

## 2 Practice

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Now apply the methods above.

**2.1** State the washer-method formula for the volume. [1]

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**2.2** For  $R = 4$  and  $r = 2$ , find  $\pi(R^2 - r^2)$ . [1]

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**2.3** Evaluate  $\pi \int_0^1 (x - x^2) dx$ . [2]

## 3 Exam-style questions

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**3.1** The washer method integrates [1]

- **A**  $\pi \int (R - r)^2 dx$
  - **B**  $\pi \int (R^2 - r^2) dx$
  - **C**  $\pi \int R^2 dx$
  - **D**  $\int (R^2 - r^2) dx$
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**3.2** The region between  $y = x$  and  $y = x^2$  (from  $x = 0$  to  $x = 1$ ) is revolved about the  $x$ -axis.

(a) Identify the outer and inner radii. [2]

(b) Set up and evaluate the volume.

[4]

**3.3** The region bounded by  $y = 4$  and  $y = x^2$  (for  $0 \leq x \leq 2$ ) is revolved about the  $x$ -axis. Set up the washer integral for the volume. [3]

## Solutions

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**2.1**  $V = \pi \int_a^b (R_{\text{outer}}^2 - R_{\text{inner}}^2) dx.$

**2.2**  $\pi(16 - 4) = 12\pi.$

**2.3**  $\pi \left[ \frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = \pi \left( \frac{1}{2} - \frac{1}{3} \right) = \frac{\pi}{6}.$

**3.1 B** — subtract the **squares** of the radii.

**3.2** (a) On  $(0, 1)$ ,  $y = x$  is above  $y = x^2$ ; outer  $R = x$ , inner  $r = x^2$ . (b)  $V = \pi \int_0^1 (x^2 - x^4) dx = \pi \left[ \frac{x^3}{3} - \frac{x^5}{5} \right]_0^1 = \pi \left( \frac{1}{3} - \frac{1}{5} \right) = \frac{2\pi}{15}.$

**3.3** Outer  $R = 4$ , inner  $r = x^2$ ;  $V = \pi \int_0^2 (4^2 - (x^2)^2) dx = \pi \int_0^2 (16 - x^4) dx.$