6.4 The Fundamental Theorem of Calculus and Accumulation Functions

Name: ______ Class: ______ Date: ______ **Total: 11 marks**

KEY WORDS accumulation function 累积函数 • fundamental theorem of calculus 微积分基本定理

Objective

Build the skills to answer exam questions on the **Fundamental Theorem of Calculus and accumulation functions** — differentiating an integral with a variable limit.

You must be able to:

- use $\dfrac{d}{dx}\displaystyle\int_a^x f(t)\,dt = f(x)$ (the **accumulation function** 累积函数)
- apply the **chain rule** when the upper limit is a function of x
- evaluate an accumulation function $g(x) = \displaystyle\int_a^x f(t)\,dt$ at given points

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

■ The Fundamental Theorem (derivative form)

If $g(x) = \displaystyle\int_a^x f(t)\,dt$, then $g'(x) = f(x)$. Differentiating "undoes" the integral. For $g(x) = \displaystyle\int_2^x \cos(t^2)\,dt$, we get $g'(x) = \cos(x^2)$ — no integration needed.

■ Chain rule with a variable upper limit

If the upper limit is $u(x)$, then

$$\frac{d}{dx}\int_a^{u(x)} f(t)\,dt = f\big(u(x)\big)\cdot u'(x).$$

For $\displaystyle\int_0^{x^2} \sin t\,dt$: the answer is $\sin(x^2)\cdot 2x$.

■ **Evaluating an accumulation function**

$g(x) = \int_1^x (2t)\,dt$. A value like $g(3) = \int_1^3 2t\,dt = [t^2]_1^3 = 9 - 1 = 8$, and $g(1) = 0$ (zero-width interval).

■ **Sign of the accumulation**

g **increases** where $f > 0$ and **decreases** where $f < 0$, because $g' = f$. Reading a graph of f tells you where g rises and falls.

2 Practice

Now apply the methods above.

2.1 If $g(x) = \int_0^x (t^3 + 1)\,dt$, find $g'(x)$. [1]

2.2 Find $\dfrac{d}{dx}\int_1^x \sqrt{t}\,dt$. [1]

2.3 Find $\dfrac{d}{dx}\int_0^{x^3} \cos t\,dt$. [2]

3 Exam-style questions

3.1 If $g(x) = \int_a^x f(t)\,dt$, then $g'(x) =$ [1]

- **A** $f'(x)$
- **B** $f(x)$
- **C** $f(x) - f(a)$
- **D** $F(x) - F(a)$

3.2 Let $g(x) = \int_2^x f(t)\,dt$, where f is continuous.

(a) Find $g(2)$. [1]

(b) Given $f(5) = 7$, find $g'(5)$. [1]

(c) If $f(t) > 0$ for all $t > 2$, state whether g is increasing or decreasing for $x > 2$, with a reason. [2]

3.3 Let $h(x) = \int_0^{2x} e^{-t^2}\,dt$. Find $h'(x)$. [2]

Solutions

2.1 $g'(x) = x^3 + 1$.

2.2 $\sqrt{x}$.

2.3 $\cos(x^3) \cdot 3x^2$.

3.1 **B** — by the Fundamental Theorem, $g'(x) = f(x)$.

3.2 (a) $g(2) = 0$ (zero-width interval). (b) $g'(5) = f(5) = 7$. (c) Increasing — $g'(x) = f(x) > 0$ for $x > 2$.

3.3 $h'(x) = e^{-(2x)^2} \cdot 2 = 2e^{-4x^2}$.